# Building and Testing
# a Cognitive Approach to the Calculus
# Using Interactive Computer Graphics

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## Acknowledgements

First and foremost I must thank my friend and mentor, Professor Richard Skemp, whose support, advice and encouragement have brought this thesis to completion. His theories have been instrumental, nay relational, in shaping my own.

I was fortunate ten years ago as a mathematics lecturer to be supported in my interest in mathematical thinking by Professor Rolph Schwarzenberger, who collaborated with me in an early article formulating ideas that led to this thesis. Subsequently the work of Professor Efraim Fischbein on the intuition of infinity and Dr Shlomo Vinner on "concept image and concept definition" helped me considerably to refine my ideas.

I gratefully acknowledge the financial support of the University of Warwick, to attend conferences of the International Group for the Psychology of Mathematics Education, and of the Royal Society, for study visits to Israel.

In writing the programs for the "Graphic Calculus" it was of great value to have the technical support of the Science Education Department technician, Ian Knight, and in testing the programs in the field I had assistance from many researchers and teachers, particularly Geoff Sheath of the Polytechnic of the South Bank, Norman Blackett at Kenilworth School and Phil Wood and his staff at Barton Peverill Sixth Form College.

I owe a considerable debt to the teachers who took part in the research, not only Norman Blackett, Kathie Lockwood and Carole Tyler who taught with the computer, but also Derek Morris, Nigel Sidley, Mrs S A Moss and George Marchi, who taught the control groups and kept diaries of their work. I record my grateful thanks to Phil Wood, Head of Mathematics at Barton Peverill College and John Clark, Head of Mathematics at Cricklade College for offering the facilities to carry out the research.

Although I have published several books and many articles, it has never been my practice to mention my family in professional publications. But on this occasion I must pay tribute to my wife and children for bearing with me as I worked long hours at home, programming and word-processing over the last six years to prepare this thesis.

To all these people, and many others who have helped me, I give my thanks.

Writing this, my second doctoral thesis, twenty years after my first in mathematics, has brought home to me forcibly the exhausting difficulties that beset those brave people who attempt to complete an advanced degree whilst having a full-time occupation. I dedicate this thesis to them: they must be very committed ... and a little crazy.

David Tall

## Tables and Figures

The tables and figures are numbered consecutively in each chapter. In the following list, figures are distinguished by being printed in *italics*.

## Declaration

I declare that the material in this thesis has not been presented in any previous thesis. Chapters 5,6 are modified versions of previously published papers (Tall [1981a, 1985e]). I acknowledge Rolph Schwarzenberger's contribution to the paper Schwarzenberger and Tall [1978], which was one of my early publications in this area of research, and the assistance of Shlomo Vinner in formulating the ideas of concept image and concept definition (Tall & Vinner [1981]).

## Summary

This thesis consists of a *theoretical building* of a cognitive approach to the calculus and an *empirical testing* of the theory in the classroom. A *cognitive approach* to the teaching of a knowledge domain is defined to be one that aims to make the material potentially meaningful at every stage (in the sense of Ausubel). As a resource in such an approach, the notion of a *generic organiser* is introduced (after Dienes), which is an environment enabling the learner to explore *examples* of mathematical processes and concepts, providing cognitive experience to assist in the abstraction of higher order concepts embodied by the organiser. This allows the learner to build and test concepts in a mode 1 environment (in the sense of Skemp) rather than the more abstract modes of thinking typical in higher mathematics.

The major hypothesis of the thesis is that appropriately designed generic organisers, supported by an appropriate learning environment, are able to provide students with global gestalts for mathematical processes and concepts at an earlier stage than occurs with current teaching methods.

The building of the theory involves an in-depth study of cognitive development, of the cultural growth and theoretical content of the mathematics, followed by the design and programming of appropriate organisers for the teaching of the calculus. Generic organisers were designed for differentiation (gradient of a graph), integration (area), and differential equations, to be coherent ends in themselves as well as laying foundations for the formal theories of both standard and non-standard analysis.

The testing is concerned with the program GRADIENT, which is designed to give a global gestalt of the dynamic concept of the gradient of a graph. Three experimental classes (one taught by the researcher in conjunction with the regular class teacher) used the software as an adjunct to the normal study of the calculus and five other classes acted as controls. Matched pairs were selected on a pre-test for the purpose of statistical comparison of performance on the post-test. Data was also collected from a third school where the organisers functioned less well, and from university mathematics students who had not used a computer.

The generic organiser GRADIENT, supported by appropriate teaching, enabled the experimental students to gain a global gestalt of the gradient concept. They were able to sketch derivatives for given graphs significantly better than the controls on the post-test, at a level comparable with more able students reading mathematics at university. Their conceptualizations of gradient and tangent transferred to a new situation involving functions given by different formulae on either side of the point in question, performing significantly better than the control students and at least as well, or better, than those at university.

# I

# Introduction

# 1. Introduction: an overview of the problem

This thesis is concerned with the building and testing of a theory of learning mathematics at more advanced levels from age 16 through to university and focuses on the subject of the calculus. It is a synthesis of a number of published papers Tall [1977 -1985] together with experimental investigations into the ideas proposed.

During this period the possible ways of introducing the ideas have changed enormously due to the arrival of the microcomputer with its fast calculations, dynamic visual displays and interactive facilities. The new technology has all the hallmarks of a change in paradigm (in the sense of Kuhn [1962]). In such a climate of change, it is more appropriate to take a wider brief and to try to forsee the kind of changes that are likely. Thus the thesis is in two parts. The first builds a theory of learning mathematics based on the cognitive development of the student. The second part puts it to the test.

The wide framework in which the subject is studied in this thesis draws together many different strands:

1. The development of a suitable framework of *educational psychology*

2. a view of the broad sweep of *historical* and *cultural* elements to set the change of paradigm in perspective and to obtain a realistic idea of what may be achieved

3. deep reflection on the possible approaches to the subject to develop a *cognitive* approach appropriate for the new technology

4. a *mathematical* analysis of the approach to confirm that it is logically sound

5. the *design* and *programming* of appropriate software

6. Empirical *educational research* into the validity of the cognitive approach and the use of the software.

Thus expertise is required in a number of different areas: education, psychology, history, anthropology, mathematics, software design, computer programming, teaching, research methodology with a leavening of philosophy and pragmatism. I shall attempt to cover aspects from all of these and, though my ability in each may be small, it is to be hoped that the whole that is created is greater than the parts.

Before the computer became a potent force in the classroom I was fortunate to have studied some of the problems encountered by students in understanding the calculus. Initial investigations (Schwarzenberger & Tall [1978]) amongst the more able students arriving at university to study mathematics revealed certain inconsistencies and conflicts that occurred with the current system of teaching the calculus. For example, students' intuitively based notion of limit included the strong belief that a limit could never be attained and that "point nine recurring" was less than one. Coupled with this belief was a use of infinitesimal ideas which are at variance with the formal definition of the real number line in analysis and could cause conflict in the understanding of a formal epsilon-delta approach to the subject.

I had long believed in the development of long-term learning schemas as described by Skemp [1962,1971,1979] and had proposed a long-term learning schema for calculus/analysis in Tall [1975]. The basic idea is to develop concepts at any stage with long-term learning in mind, and to reduce cognitive strain brought about by inappropriate learning at earlier stages. Such an approach must take account of two important factors.

The first is specific to the calculus: that the majority of students taking the subject will not continue to study mathematical analysis. In a recent review (DES [1982]) 6% of those studying A-level mathematics carried on to a degree course in mathematics, statistics and computing where they might meet mathematical analysis at a more rigorous level, 38% entered various science degree courses where they might use the subject in applications, 15% entered teacher training and the remainder went into employment or other courses where calculus was likely to be of less value. Thus the calculus curriculum must serve a variety of purposes: it must be a valid end in itself, a suitable preparation for practical calculus in experimental, theoretical and social sciences, as well as being part of a long-term schema for analysis.

The second factor is a general educational concern. When a concept is met early in the curriculum the learner may lack the sophistication to appreciate it in a wider context. It may only be understood within the context of the learner's current cognitive development. This will lead to a limited view of the concept, which may contain factors which will conflict with a wider meaning of the concept met at a later stage. Thus a long-term learning schema must take into account cognitive development as well as the mathematical development. It must also identify points which may involve conflicts in learning and offer ways of assisting the learner to reconstruct his (or her) mental concepts.

I was fortunate at this stage to work with Shlomo Vinner who had coined the terms "concept image" and "concept definition" to differentiate between the mental ideas associated with a concept and the form of words used to describe the concept. Based on these terms I re-interpreted the results found in investigating calculus concepts and wrote Tall & Vinner [1981]. The ideas in this paper, and their ramifications, will be discussed in chapter 3 of this thesis.

Ausubel et al [1968] describe the notion of advance organiser, which is introductory material at a higher level of generality related both to the task and to the learner's cognitive structure. In a complementary fashion, after the style of Dienes

[1960], this chapter introduces a generic organiser which allows the learner to form a global gestalt for a higher level concept by exploring examples of that concept. Computer software can provide an ideal environment for such exploration. The use of generic organisers is part of a cognitive approach to mathematics, giving the learner appropriate experiences so that (s)he is cognitively ready for new mathematical concepts as they are introduced.

An individual's concept imagery depends on experience and internal mental processing, giving the possibility of evoking different imagery in different contexts, with possible conflicts arising between various aspects of the concept. All individuals have their own concept imagery, and that includes pupils, teachers, mathematics professors, the readers of this thesis and, not least, its author. Interpretations of what constitutes the "truth" in mathematics are coloured by our own concept imagery. Skemp [1979] writes of three modes of reality testing: with reference to the world outside, by communication with others and by personal reflection on the internal consistency of our theories. All three modes are relative, not absolute. We are comforted to be part of a wider culture which shares some of our beliefs, but that does not of itself constitute a "proof" of the accepted truth. In particular, if the intuitive beliefs of pupils do not coincide with the accepted beliefs of the sophisticated mathematical culture, that does not render them invalid. A number of my recent papers have been devoted to demonstrating that alternative views are possible. (Tall [1979a,1980b,1981b].)

The calculus is rich with different possible interpretations. It has developed over the centuries and various schools of thought have aspects of their viewpoints persisting in our current culture. For instance, the calculus of Leibniz, with its infinitesimal notation dx,dy persists in an epsilon-delta context in which the expression dy/dx is regarded by some as a ratio and by others as a single indivisible symbol.

Wilder [1968] writes potently on the manner in which cultural elements in mathematics are dispersed. New ideas are diffused into a culture; there may be a considerable "cultural lag" before they are widely accepted, perhaps being met by "cultural resistance" which may lead to further delay or even rejection. Diverse elements that persist in the calculus can provoke heated arguments and strong differences of opinion. As they concern the concept imagery of the individual, suggestions of alternative ways of looking at the subject may be interpreted by an individual as an attack on his self-image, leading to powerful resistance.

To help provide the context for an objective view of various interpretations of the calculus, I have found it helpful to review important differences of opinion in the history of the calculus from a cognitive viewpoint (chapter 4).

In chapter 5 I shall consider various possible approaches to the calculus from a wide range of choices now available e.g. Tall [1981a, 1985], Tall & West [1985], Winkelmann [1984a,b], Lane [1985], Stoutemyer [1985], Hodgson [1985]. For mathematical and cultural reasons some approaches are considered less sound than others and I shall analyse their validity and some of the difficulties that they present to the learner.

In Chapter 6 I shall introduce a cognitive approach to the calculus (Tall [1985a-d]) using interactive computer graphics which takes into account the two main requirements for a long-term learning schema mentioned earlier. The software consists of a sequence of programs designed as generic organisers for various concepts and processes in the calculus.

The need for the calculus to be both an end in itself and a precursor of formal studies is met by giving the ideas an immediate relevance but sowing the seeds for future study where that is appropriate. The relevance of the ideas to the current development of the learner is approached by giving the concepts an immediate global gestalt.

The human brain is particularly well adapted to process *visual* information. It also functions in real time and is extremely good at handling ideas presented *dynamically*. Thus the programs are designed for demonstration of dynamic visual representations of the concepts. Allowing for the wide variety of concept images that different individuals may form when confronted with a given experience, it is also important for the individual to be able to *explore* ideas to fill out their own imagery. The same programs are designed to allow interactive investigation. Thus a student may *investigate* the gradients of the graphs of $x^2$, $x^3$, *conjecture* the formula for the gradient of $x^n$ and *test* it for various values of n, such as n=5, 33, -1, 1/2 and so on. The approach is intended to give a coherent concept image of the geometrical meaning of the derivative of $x^n$ suitable as an end in itself or as the basis of further study. It allows the student to learn the *processes* of mathematical investigation rather than just the *results* of mathematical theory.

The testing of the theory begins in chapter seven with a discussion of methodology. After preliminary investigations in a local school, a more formal framework was organised with the researcher working with a class of students in the same school and a second class acting as a control. In another school two more classes tried out the experimental approach without the researcher being present and four other classes acted as controls. The larger number of control students enabled a matching to be carried out for more in-depth statistical study.

A third school also offered to take part, but here the experiences were less happy. Chapter eight describes the experiences in the various classrooms and the attitudes of the students towards the use of the computer.

A pre-test and post-test were given, together with two brief questionnaires studying the growth of the concept images of gradient and tangent. The next four chapters review the students' responses to these investigations: their ability to carry out mathematical processes, their responses to open-ended questions about the concepts, their use of language, and their notions of gradient and tangent.

It is found that the experimental students are far better at visualizing the gradient of the graph and at responding to questions about gradient and tangent in a new situation.

Chapter 12 considers the conclusions and open questions that are ripe for future research. At the time of writing there are individuals who have experience of various aspects of the learning of the calculus: through dynamic graphical representations, numerical algorithms and symbolic manipulation packages that perform automatic symbolic differentiation and integration, but there is much work to do in producing an appropriate synthesis of these ideas. The paradigm is changing. The climate is ripe for investigation and research.

# II

# Building

# 2. Review of investigations into the teaching and learning of the calculus

## Teaching and learning

There have been many articles on the teaching of the calculus, but most of these are concerned with the nature of the subject matter and the manner in which it is taught, with very little emphasis on the act of learning. The idea of studying the psychology of learning the calculus is a fairly recent venture.

In the collection of readings on the calculus selected over a sixty year period from the *Mathematics Teacher* by Grinstead & Michaels (eds.) [1977] only three out of forty one articles chosen were placed in the section entitled "Pedagogical Overview". Of these one is a brief consideration of objectives in a calculus course and another is a short analysis of calculus problems suitable for a student entering college to read science or engineering. The articles outside this section mainly concern the history of the subject and various aspects of the calculus curriculum. Some are classified as being pedagogical, such as a discussion of the meaning of a geometric tangent or the motivation for the chain rule, but these are concerned with the technicalities of definitions and proofs, with no mention of student reaction.

Only one article by Cummins [1960] can be deemed to be directly concerned with the act of *learning* the subject as opposed to *teaching*. Here the students used an "experience-discovery approach" which included "materials ... to develop understanding in the use of some of the fundamental ideas before these concepts were subjected to critical discussion" , "one section ... devoted to developing the calculus as a deductive system" and "a series of study-guide sheets ... by which students, either independently or with the help of class discussion, could arrive ... at some methods and facts of the calculus."

The students were tested before and after and compared with a control group. In traditional skills they scored at exactly the same level as the students taking a standard course, but significantly higher on a questions relating to the fundamental theory and logical relationships between parts of the calculus. For example the students were asked to give an explanation for one of the quotient, product or chain rule for differentiation; 25 out of 38 experimental students were successful, but only one out of 24 control students. When it came to explaining the logical connection between differentiation and integration, 18 experimental students were successful compared to none from the control group. When questioned about the new approach, 22 out of 53 control students responded that they understood the *reasons* for doing, rather than just the mechanics.

Periodically there are articles in the *Mathematics Teacher* on other aspects of the teaching of the calculus. But apart from those on specific items in the calculus syllabus these seem to concern such topics as the feasibility of teaching calculus (e.g. Kinney [1923], Cosby [1923], Farmer [1927], Kucinski [1959]) and whether the calculus should be taught in a precise logical fashion or with appeal to intuition and physical ideas (e.g. [Taylor 1962]).

In Britain the position is much the same. (See, for example Orton [1980, 1985] who cites a number of reports on the teaching of the calculus.) A significant difference is that the British Educational system allows the calculus to be studied at an early age for examination at 16+, though this continues to be a topic of some controversy.

**The Second International Mathematics Study**

The Second International Mathematics Study is concerned with collating in-depth cross-sectional information about the teaching and learning of mathematics at school level in over twenty nations. The prime focus is the mathematics content:

(1) the expected content: the "Intended Curriculum"

(2) the teachers priorities in teaching : the "Implemented Curriculum"

(3) the students mastery of the mathematics: the "Attained

Curriculum".

The intended curriculum was obtained by collecting responses from a survey in each country to ensure comparability. The implemented curriculum was studied through school and teacher questionnaires and the attained curriculum through student questionnaires.

The items relating to the calculus from the study in Canada are discussed by McLean et al. [1984] This revealed that on average 60% of class time was spent with the whole class working as a group, 11% in small group instruction and 27% with the students working individually. The main reasons for covering a topic was because it was in the Curriculum Guide or because it was useful in later work. In a factor analysis of teacher opinion the most significant factor was the belief that secondary-level calculus should focus on applications and intuitive understanding and should not follow the university syllabus. However the report comes to the conclusion that

> the reports of topic coverage and teacher beliefs support the view that the Ontario Calculus Course may best be viewed as an introduction to a university calculus course.

The survey was taken in 1981-2, at which time:

> ... teachers rely upon their own prepared materials and the student textbook and make little if any use of recent

innovations in electronic technology to aid in their teaching of Calculus. The prime determinant of topic coverage and teacher practice appears to be the student textbook. Attempts to institute change in current practices would appear dependent upon modification of these student textbooks.

The questions in the student test were selected on four "behavioural" levels – computation, comprehension, application and analysis. Close inspection shows that they reflect the current teaching practices at school and university in the U.K. and do not delve deeply into the student's cognitive development. However, the study is most useful for relating student achievement to the effects of teaching.

### The case of non-standard analysis

In 1976 a new approach to the calculus appeared as a college text-book by Keisler [1976] with a research study of its effectiveness by Sullivan [1976]. The method of approach was a combination of axioms for the real numbers and for a larger set of hyperreal numbers containing infinite and infinitesimal elements, together with a pictorial interpretation of these objects using microscopes and telescopes. Sullivan's investigation compared the students' response with a control group both in standard and non-standard analysis. She found that the experimental group scored at least as well as the control group in standard methods (defining basic concepts, computing limits, proofs, and applications) but were better at those aspects of the course with a non-standard flavour. (The classic example was their ability to explain why a certain example was not continuous, a statement which looks easier in non-standard terms than standard, because the standard version has more quantifiers.) Sullivan's study is not a conceptual one, but a comparison between two different mathematical approaches. It gave favourable performances in tests by the students and favourable comments from students and lecturers yet failed to become an established teaching method in the wider mathematical community. We shall return to this point in chapter 4.

## Conceptual studies

Until recent times there have been few studies of the psychological nature of the concepts essential to the calculus. Taback [1975] studied the limit process in young children aged 8 to 12 in a Piagetian framework of concrete and abstract levels of thought.

Orton [1977] considered misconceptions arising from the method of obtaining gradients using the tangent as a limit of chords or secants. One method is to draw the chord PQ as a line segment from P to Q on the graph and to state that as Q moves to P through successive positions $Q_1$, $Q_2$, $Q_3$, the chord PQ approximates more nearly to the tangent at P, so that the tangent PT is the limiting position of the chord when Q coincides with P (figure 2.1).

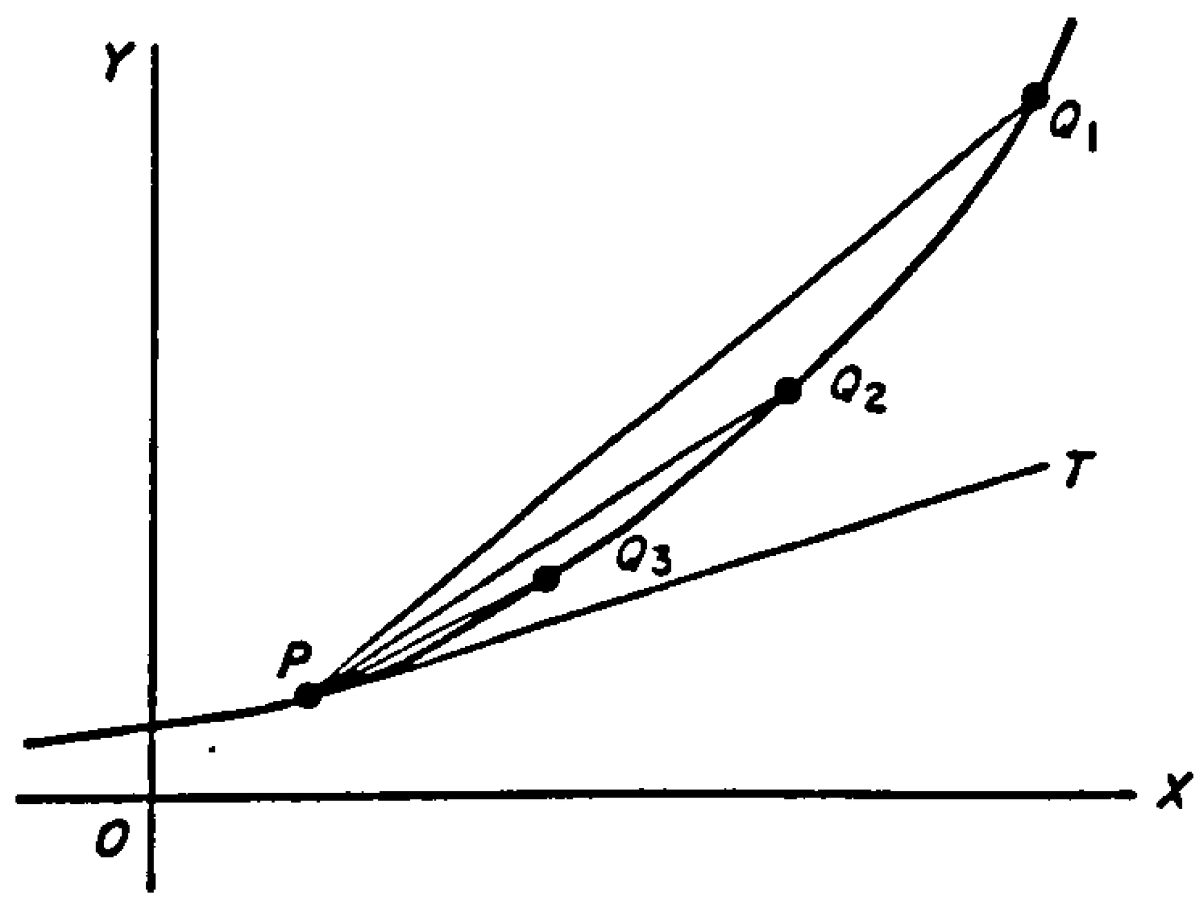



Figure 2.1

The author remarks

> This seems to me to be the approach most open to criticism. My experience with pupils in school was that some of them could only appreciate that the chord disappeared to a point and could not see how the tangent mysteriously appeared.

He goes on to comment that, even if the chords are extended as secants, then 17% of pupils and students of a sample in the 16-21 age group still wanted to say that the line disappeared or became a point. A further 21% could only state that the line became shorter or that a smaller and smaller area was enclosed.

Conflicts experienced by students in limiting processes and infinite decimals were reported by Tall [1977] and Schwarzenberger & Tall [1978]. In a questionnaire given to students arriving at university, many responses reported that nought point nine recurring is considered as less than one. Yet nought point three recurring could be considered as precisely a third. By multiplying the latter by three we arrive at a conflict.

One may conjecture that the limiting object is attributed the same properties as the objects tending to the limit. Thus 0.9, 0.99, 0.999 is an increasing sequence whose terms never reach 1, so they "tend to nought point nine recurring" and this limit object does not equal one. This phenomenon occurs with more general limits of sequences. The phrase "gets close to" suggests that the limiting objects are "not equal to" the limit, and the standard examples given almost always have this property. So it is possible to believe the property that if a sequence tends to a limit, then the terms of the sequence can never reach that limit.

I shall term a limiting object that is conceived as having the same properties as the objects in the limiting process a *generic limit*. It need not be a limit in the mathematical sense. Thus a generic limit of the sequence 0.9, 0.99, ... is "nought point nine recurring", visualized as being "just less than one", whilst the mathematical limit equals one.

Historically the notion of generic limit is enshrined in Leibniz's "principle of continuity" (see chapter 4) and in Cauchy's notion that the limit of continuous functions is continuous. The phenomenon occurs when the rules which apply in one context are implicitly assumed in a broader context, such as the nineteenth century "Principle of permanence of algebraic laws" which asserted that an extended number system would obey the normal rules of arithmetic.

The publications of Fischbein [1978] and Fischbein et al. [1979,1980] are concerned with the intuition of infinity and its inherently contradictory nature. He cites the conflict between the intuition of the single potential infinity and the many infinities of cardinal number theory as one example, and the conflict between the finite number of points that may be marked physically on a line compared with the infinite number of points that are theoretically possible.

In Tall [1980b] I discuss how these conflicts might result from extrapolating finite experiences in different ways. For example, extrapolating one-one correspondences gives cardinal numbers in which addition and multiplication are possible but not subtraction or division, whilst extrapolating measuring experiences gives a kind of mathematics akin to non-standard analysis, admitting a full arithmetic including infinites and infinitesimals.

Non-standard analysis allows $s_n=1-1/10^n$ (nought point nine to n places) to have a value for infinite n. In this case $s_n$ is infinitesimally smaller than 1 and gives a mathematical meaning similar to the cognitive idea of a generic limit. (See the postscript to Tall [1981d].)

However, the two concepts are not the same. Students often view the generic limit as "being as close to 1 as is possible without actually being equal to it". The non-standard number $1-1/10^n$ for infinite n does not have this property because $1-1/10^{n+1}$ is closer to 1 and still not equal to it. Thus non-standard analysis does not provide an exact match for the cognitive concepts.

However, non-standard analysis does provide an alternative formal context in which some aspects absent from the standard theory (e.g. existence of infinitesimals) become admissible. This shows that "standard analysis" can only provide a *relative* context in which to judge the student response, not an *absolute* one. It is my firm opinion that the reasons underlying the learner's thinking are of paramount importance in the psychology of learning and merit investigation for their own intrinsic value rather than judgement by a particular mathematical standard (Tall [1980b/c/d, 1981b]).

I have elsewhere pointed to instances where mathematical theories could cause the misinterpretation of psychological events, for instance Tall [1980b, 1981b] indicates that the theory of cardinal numbers is an inappropriate context for certain intuitions of infinity and Stewart & Tall [1979] suggests that Piaget's use of the set-theoretic definition of cardinal numbers misleads him into neglecting the role of counting numbers. One might also quote the mesmeric effect that group theory had on Piaget in his attraction for equilibrium in the psychological growth of the individual, leading to inconsistent usages of the theory in his idea of "groupings". These ideas are worthy of further study and will be discussed in following chapters.

**The thesis of Orton**

The first major body of work on the study of concepts in the calculus is due to Orton [1977, 1980a/b, 1983a/b, 1985].

In his thesis, Orton [1980a] selected sixty students from four mixed secondary schools to represent a typical spread of ability and fifty college students undergoing teacher training. In both groups there were students from all years of study and the complete sample contained 55 males and 55 females. A considerable number of items on differentiation and integration were presented to each student during two clinical interviews. Following the interviews the items were subdivided and regrouped under 38 headings.

The data was handled globally by marking each of the 38 headings out of 4 (with 0 representing the absence of a worthwhile response and 4 representing a completely satisfactory answer) and performing a factor analysis. The main factor in differentiation is given as

> a general intellectual/educational factor reflecting understanding of rate of change as a simple ratio and of derivative as rate of change but with no involvement with substitution or with limits" with 17.4% of the variance.

Other factors are:

> average rate of change and instantaneous rate of change on a curve by substitution (9.0%)
>
> the differentiation process and symbolism (8.3%)
>
> Applications of differentiation to gradients and stationary points on graphs (6.8%)
>
> Limits of number sequences (6.7%)
>
> Elementary rate of change (5.4%)
>
> (Orton [1980b] p.208)

The interpretation of this number crunching exercise and its value in improving the teaching of the calculus is hardly revealing.

Another analysis re-classifies the scores 1-4 on each item using a Piaget style grading into late concrete (2B), early formal (3A), late formal (3B), according to the style of thinking that seemed to be necessary to reach any given level. By converting the latter grading back to a numerical scale to give a "measure of cognitive demand", a statistically significant correlation between the original scores and the cognitive demand was achieved. The statistical significance shows a correlation between the table of numbers 1-4 and the ranking 2B-3A-3B whilst saying nothing about the validity of the "Piagetian levels" which the author bases on his own interpretation of the students' performances. Nevertheless, an idea of cognitive demand of each item is of great importance and could be of value in identifying those areas of greatest conceptual difficulty. Orton [1980b] states that the items of greatest cognitive demand are:

rate, average rate and instantaneous rate
instantaneous rate of change and tangents
differentiation as a limit
the use of the delta symbolism

limit of sequence equals area under graph
explaining integration
integral of sum equals sum of integrals
volume of revolution

relationship between differentiation and integration.

The later publications of Orton [1983a/b] omit reference to factor analysis and Piagetian levels, turning instead to a classification of the errors as *structural*, *executive* or *arbitrary*. Structural errors arise "from some failure to appreciate the relationships involved in the problem or to grasp some principle essential to the solution", executive errors "involve failure to carry out manipulations, though the principles may have been understood" whilst arbitrary errors are "those which the subject behaves arbitrarily and fails to take account of the constraints laid down in what is given". Both papers are largely concerned with describing a selected number of tasks and the students responses using this error classification, which the author finds "more difficult than anticipated, though some errors were clearly of one type only".

There is some interest in the errors which occur, for example the difficulty that student have in carrying out what might be considered routine algebra. This is certainly at a level which should cause some concern.

The thesis of Orton [1980a] constantly shows the mathematical thinking of the students being judged by the conventions of a particular approach to mathematics. For instance the question:

> Explain the meaning of ... dx

is only adjudged correct if the answer is of the form

> Not usually meaningful, but may be thought of as 'with respect to x'

There is a school of thought that *defines* dx as any increment in x and the corresponding increment in y to the tangent as $dy=f'(x)dx$ and this definition occurs in school texts (see, for example, Quadling [1955]). By selecting one view of calculus the author denies the validity of an alternative interpretation.

On page 320 the author concludes:

> The symbols which caused the greatest problems to students were dx and dy. This was expected in the sense that the symbols are not really meaningful except when used together as dy/dx or when used in integration, for example $\int f(x)\,dx$. Four main types of incorrect response were apparent. Twelve school students and 17 college students explained dx as "the differential of x" or "the differentiation of x" or "the rate of change of x ... "

Here one response that may be regarded as correct is denied and classified with other errors that may be of a different nature.

Another task where the psychological nature of limits is important occurs with the "staircase with treads" where extra half-size treads are inserted between each tread, then the process repeated successively with treads half this size again. (Figure 2.2.)

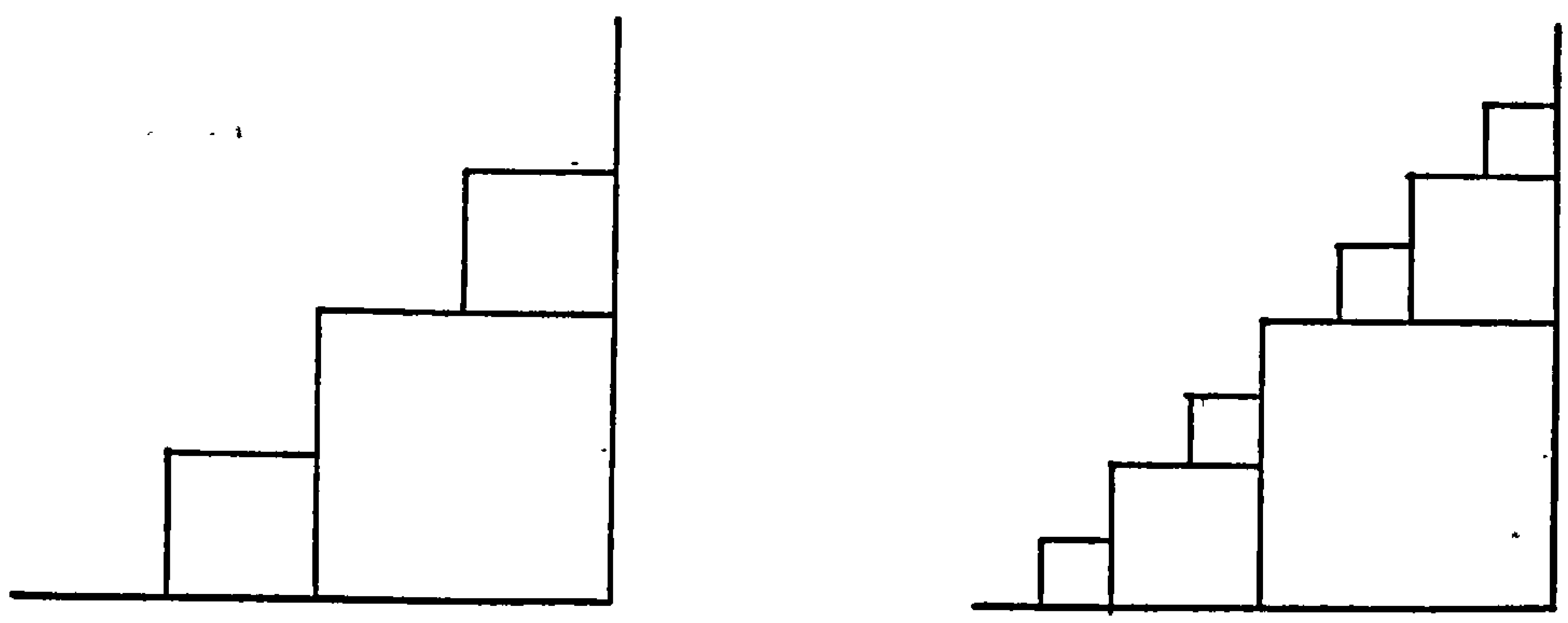


Figure 2.2

The questions posed are

(a) If this procedure is repeated indefinitely, what is the final result?

(b) How many times will extra steps have to be placed before this "final result" is reached?

(c) What is the area of the final shape in terms of "a", ie what is the area below the "final staircase"?

The interview includes the question:

> Can you use this formula to obtain the 'final term' or limit of the sequence ?

which the author justifies by the statement:

> The expression 'final term' was again used in an attempt to help the students understand the meaning of limits.

This terminology seems bound to sow the seeds of psychological conflict. In non-standard terms the graph for infinite n is a staircase with an infinite number of steps of infinitesimal tread. So "generic limit concepts" are quite reasonable in a non-standard theory whilst being "structural errors" in a standard context.

On page 104 the question

> Do you know any other method of obtaining areas under curves?

is allowed responses in the form

"Yes, integration" or "Yes, integration is derived as the sum of an infinite number of rectangular strips under or enclosing a curve, each strip being of negligible or infinitesimal width", or equivalent.

The use of the terms "infinite" and "infinitesimal" again needs sympathetic interpretation.

On page 348 the author provides suggestions for action which contain one statement likely to cause cognitive conflict and another which is mathematically incorrect (italicised in the following extract):

> A study of the relationship between average rate and the gradient of a line through two points on the curve comes first. The points may then be taken closer and closer together, so that the limiting case, *when the points coincide*, may be considered. If extended chords or secants, and not simple chords, are used, the notion that the line still exists, that it touches *but does not cross* the curve, that this is what we mean by the tangent to the curve, and that the gradient of this tangent is a measure of the rate of change at that point may all be introduced.

The idea that points get closer until they coincide causes cognitive conflict because, when the points coincide there is only one point through which the "chord" passes and this no longer specifies the line. The statement that a tangent touches, but does not cross, is false, as a look at a point of inflexion will confirm.

Schwarzenberger & Tall [1978], Tall & Vinner [1981], Tall [1985f] all give evidence that the conventional presentation of mathematics itself causes cognitive conflict. If this is true then the analysis of student errors in conventional terms cannot give illumination without an analysis of conflicts in the mathematical presentation itself.

Orton [1985] cites the report of the Mathematical Association (1951):

> ... the early development must be gradual; any rushing of the introduction will lead to chaos.

The advice to take the early stages of the calculus slowly occurs often in the literature. *But there is no guarantee that taking a teaching method slowly will improve things if that method is wrong.*

Orton gives no remedies for solving the problems of algebraic manipulations in [1983a/b]. To improve the conceptual teaching of calculus he suggests (following Neill [1978]) that the gradient at a point P on a curve may be studied through the gradients of the secants using an electronic calculator and the area under the graph may also be calculated numerically. These suggestions are repeated in greater detail in Orton [1985].

## The thesis of Cornu

Cornu [1981, 1983] studies the conceptual development of the limit process from school to university. He begins with a questionnaire in school that is intended to find colloquial uses of the term "limit" before it is met mathematically. He is interested in how these colloquial meanings may effect the mathematics. For instance the notion of limit may be something like a speed limit, which cannot be passed, so the colloquial meaning of the term may colour the technical meaning. He then designs a questionnaire to investigate the idea of limit at university and shows how a *conception spontanée* may act as an *obstacle* to developing the technical meaning of a *conception propre*. He links these terms to the theory of concept image and concept definition of Tall & Vinner [1981] which is discussed the next chapter of this thesis.

He discusses the notion of "obstacle", introduced by Gaston Bachelard in "La formation de l'esprit scientifique" and further discussed by Guy Brousseau:

> "Un obstacle est une connaissance: il fait partie de la connaissance de l'élève. Cette connaissance a en générale

été satisfaisante à une certaine époque, et pour résoudre certains problèmes. C'est précisément cet aspect satisfaisant qui a ancré la connaissance et en a fait un obstacle. Cette connaissance devient inadaptée, car on se trouve face à des problèmes nouveaux; mais cette inadéquation peut ne pas etre apparente.

[An obstacle is a piece of knowledge; it is part of the knowledge of the student. This knowledge has in general been satisfactory at a certain time for solving certain problems. It is precisely this satisfactory aspect which has anchored the knowledge in the mind and made it an obstacle. The knowledge proves to be inadequate when faced with new problems and the inadequacy may not be obvious.]

Cornu outlines four major obstacles to the understanding of the notion of a limit which parallel difficulties in history:

1) The metaphysical aspect of the idea:

Les mathématiques ne se réduisent plus à des calculs, à des propriétés algébriques simples. L'infini intervient, et il est entouré de mystère. L'élève a du mal à "y croire":

"ce n'est pas très rigoureux ... mais ça marche"
"ça n'existe pas ... c'est abstrait".

... Cet obstacle rend difficile la compréhension de ce que peut etre la limite d'une suite, surtout lorsque cette limite ne peut pas etre calculée directement par des mèthodes algébriques. Comment etre sur qu'un nombre existe, si on ne peut pas le calculer?

[The mathematics no longer reduces to calculations and simple algebraic properties. Infinity intervenes, and it is surrounded in mystery. The student has the misfortune to believe:

> "It's not very rigorous ... but it works"
> "It doesn't exist ... its abstract"

...This obstacle makes the understanding of what might be the limit of a sequence difficult, especially when the limit cannot be directly calculated using algebraic methods. How can we be sure if a number exists if we can't calculate it?]

2. The infinitely small and infinitely large

tout se passe comme s'il existait des nombres très petits, plus petits que les "vrais" nombres, mais cependant non nuls. Le symbole $\varepsilon$ contient chez beaucoup d'élèves une signification de ce type. $\varepsilon$ est plus petit que tout nombre réel, mais n'est pas nul. De façon analogue, il semble exister un entier plus grand que tous les autres, mais cependant pas tout à fait infini.

Quelques exemples:

"... la plus juste possible"

... "les pentes sont à un certain moment très peu différentes mais pas égales"

"le plus grand nombre, c'est 0,999... :c'est le dernier nombre avant 1".

[Everything happens as if there exist very small numbers, smaller than "real" numbers, but nevertheless not zero. The symbol ε represents for many students a symbol of this type, ε is smaller than all real numbers, but not zero. In a similar fashion there seems to exist an integer larger than all the others, but nevertheless not wholly infinite.

Some examples:

> "... the most accurate possible"
>
> ... "the slopes are, at a certain moment, very little different, but not equal"
>
> "the largest number is 0.999... : its the last number before 1".]

3. Is the limit attained?

Le débat pour savoir si telle limite est atteinte ou non se retrouve chez les élèves ... Certaines élèves emploient des expressions différentes selon que la limite est atteinte ou non. Par exemple, "se rapproche" et "tend à se rapprocher". Ou encore, "tend vers" est réservé aux cas ou on n'atteint pas la limite.

[The discussion to know if a true limit is reached or not often occurs with students,... Certain students use different expressions, according as to whether the limit is reached or not. For example "approaches" and "tends to approach". Once again, "tends to" is reserved for the case where we don't reach the limit.]

4. The passage to the limit

Un autre obstacle nous semble important: il s'agit du problème du passage du fini à infini. Nous l'avons déjà vu, les élèves ont tendance à isoler "ce que se passe à l'infini". Par exemple, dans l'activité sur la tangente, "la régle va tomber". Ou encore, des valeurs approchées sont prises "en vrac", sans idée de "se rapprocher de". Il s'agit d'une vision statique, qui fait obstacle à une vision plus dynamique, dans laquelle ce qui se pass "dans le fini" permet de prevoir ce que se passe "à l'infini", et donc de parler de limite.

Mais ce passage du statique au dynamique doit à son tour etre suivi d'un passage du dynamique au statique, ou le statique est cette fois celui de la définition quantifiée de la limite, définition ou rien ne bouge: on se donne ε etc... ... Ils correspondent au passage de la notion de valeur approcheée à celle de valeur approchée "aussi bien qu'on veut", puis à la notion "$\forall\ \varepsilon>0\ \exists\ N$ ...".

L'acquisition de la notion de limite nécessite le franchissement d'autres obstacles: inégalities, conditions suffisantes, valeur absolue, passage de la convergence monotone à la convergence, etc ... Mais ces obstacles ne sont pas propres à la notion de limite; ils lui sont extèrieurs.

[Another obstacle seems important to us: the passage from the finite to the infinite. We have previously seen that students tend to place concepts involving the "passage to infinity" in a separate category. For example, when dealing with the tangent, "the ruler [through coincident points] falls down". Or, the limiting values are taken approximately, without any idea of the limiting process. It is a question of a static image which causes an obstacle to a more dynamic perception, in which what happens "in the finite process" allows us to see what happens "at infinity" and then to speak of a limit.

But this passage from static to dynamic must, in its turn, be followed by a passage from dynamic to static, where the static, this time, is that of the quantified definition of the limit, a definition where nothing moves: we are given $\varepsilon$ etc ... These correspond to a passage of the notion of a value approaching "as close as we desire", then the notion "$\forall\ \varepsilon>0\ \exists\ N$ ... "

The acquisition of the notion of limit necessitates the clearing of other obstacles: inequalities, sufficiency conditions, absolute value, passage from monotone convergence to convergence, etc... But these obstacles are not part of the notion of a limit, they are separate from it.]

In the light of these obstacles, he suggests a teaching sequence in which pupils explore the notions and discuss their own ideas of limits when they meet the formal definition.

The thesis of Robert

Aline Robert [1982] is concerned with the concept of limit of numerical sequences at university level. She restricts her study to a single questionnaire on convergence of sequences given to a cross-section of 1253 students over a wide range of higher education. Every aspect of the questionnaire is considered in minute detail. For instance the responses to the definition of a convergent sequence are classified as follows:

Les modèles primitifs (stationnaire, barrière, monotone)
[Primitive models (stationary, barrier, monotonic)]

Les modèles dynamiques (dynamique, dynamique monotone, "tends vers")
[Dynamic models, (dynamic, monotone dynamic, "tends to")]

Les modèles statiques (statique, préstatique)
[Static models (static, pre-static)]

Le modèle mixte
[mixed model]

Les copies sans modèle exprimé (tautologie, définition, pas de modèle exprimé)
[responses without an explicit model (tautology, definition, no explicit model)]

The classification results from a close scrutiny of the written responses to see if there are any implied mental models. For instance primitive models have responses such as

stationary: "Les derniére terms ont toujours la même valeur

...

["The final terms always have the same value"]

barrier: "Ses valeurs ne peuvent pas dépasser 1"

["The values cannot pass 1"]

monotonic: "Un suite est convergente si elle est croissant majorée (ou décroissant minorée)"

["A sequence is convergent if it is increasing and bounded above (or decreasing, bounded below)"]

Dynamic models include

"les images se rapprochent d'un nombre de plus en plus près"

["The values approach a number more and more closely"]

or have a sense of movement implied by phrases such as "tend vers" ["tends to"].

Mental models are classified as static if the respondent reformulates the standard definition in a personal way:

static: "Tout intervalle contenant 1 contient tous les $u_n$ sauf un nombre fini"

["All intervals contain all the $u_n$ except a finite number"]

or, with slightly less precision,

prestatic: "Les eléments de la suite finissent par se trouver dans un voisinage de 1"
["The elements of the sequence end up by being found in a neighbourhood of 1"].

Mixed models juxtapose static and dynamic in one response.

The final class includes all those responses which do not explicitly indicate a mental model. They may be tautologous:

"Un suite convergente est une suite qui a une limite (finie)"
["A convergent sequence is a sequence which has a (finite) limit"]

or the *exclusive* use of the formal definition

"Pour tout $\varepsilon$ il exist N tel que pour tout $n>N$ $|u_n-1| < \varepsilon$"
["For every $\varepsilon$ there exists N such that for all $n>N$, $|u_n-1| < \varepsilon$"]

or they may be unclassifiable.

Mme Robert traces the persistence of these models through the years of higher education, treating all the questions of her questionnaire with the same attention to detail. In her conclusion she addresses herself to the problem of correcting the errors (page 430):

> Nous avons vu que les origines des différentes erreurs ne sont pas simples et qu'en tout cas, plus la représentation exprimée de la convergence des suites ressemble avec des mots à la définition mathématique, meillueres sont les procédures. Il est donc tentant de proposer un travail spécifique, explicite, sur ces représentations exprimées et desinées, quelque temps après le cours, pour faire prendre conscience aux étudiants de leur images mentales, en essayant de rectifier de manière métamathématiques, par une réflexion sur ces images mentales et ces dessins, les répresentations erronées (primitives essentiellement).
>
> [We have seen that the origins of different errors are not simple in every case, the more the explanation of the convergence of series (in words) resembles the mathematical definition, the better the procedures. It is best, therefore, to set a specific task, explicitly relating to these explanations and diagrams, sometime after the course, to make the students conscious of their mental images and to try to rectify them in a mathematical way, by reflecting on their images and pictures of these erroneous (essentially primitive) interpretations.

She acknowledges the wide variety of difficulties occurring in the theory of limits and, in line with Cornu after her, finds conflicts and obstacles in learning. To improve matters she suggests that the students should reflect on their mental images and compare them with a wide variety of mathematical examples to help rationalize their understanding.

**Other relevant research**

The major obstacles found in these French investigations are very much in tune with those of Schwarzenberger & Tall [1978] and Tall & Vinner [1981].

The problem with "nought point nine recurring" was studied again in Hebrew by Kidron & Vinner [1983], producing the same kind of phenomena. It has now occurred in studies in three different languages: English, French and Hebrew.

In the USA, Hubbard & West [1985] have developed a computer drawing approach to differential equations. Although they have not done a controlled research study, their experience shows a variant of the metaphysical problem of "existence" in the solution of differential equations:

> Even after a term of studying such [existence and uniqueness] proofs, the students still do not really know

the difference between the statements:

"This equation has no solution"

"This equation cannot be solved in elementary terms."

Because the students cannot find the formula for the solution, they do not believe the solution exists.

Thus the available research shows both cognitive and mathematical obstacles to the study of calculus which may be persistent and difficult to resolve. They originate in a variety of ways. Some of them are verbal in nature, due to the (necessary) informal way of introducing the ideas, some are due to generic limit concepts, extending implicit generic properties in an inappropriate way to a broader concept, some are due to the inherent difficulties of the subject: the mathematical nature of the limiting process, the area as a limit of a sum, and so on.

One may attempt to correct some of the errors by explicit teaching (as occurred with errors in algebra in the Chelsea program "Strategies and Errors in Secondary Mathematics"). But there are many factors to be considered at any one time and gains in one place may upset the delicate balance elsewhere. Rather than treat each "error" in isolation, this thesis will propose a coherent approach to the theory, balancing cognitive and mathematical development. The next chapter will concentrate on psychological theory, followed in subsequent chapters by a review of cultural and mathematical aspects, before embarking on specific suggestions for a cognitive approach to the calculus.

# 3. Psychological notions contributing to a cognitive approach to mathematics

In this chapter I shall outline certain areas of psychology which help us to understand and predict the mental processes in mathematical learning and thinking. Behaviourist theories are inappropriate here because they deal with the surface structure of stimulus-response behaviour and fail to explore the deep structure of mathematical thinking. Of far greater value are theories of "meaningful" cognitive psychology, linking cognitive growth to the development of a knowledge domain. I shall consider essential ideas relevant to my study from several major theorists before making my own contribution.

My main aim is to develop an approach to learning and understanding mathematics in a manner that takes into account both the cognitive growth of the learner and the structure of the mathematics. For this reason I shall specifically seek theoretical viewpoints which explain the positive and negative sides of the learning process.

Difficulties are caused by cognitive obstacles such as the need to reconstruct one's cognitive structure in the light of new information. Here I shall introduce the ideas of "concept image" and "concept definition" to distinguish between the total cognitive structure associated in the mind with the concept and the formal description of the concept. The concept image governs our intuition of the concept, and I shall put forward the thesis that we can better improve our understanding of mathematics through encountering experiences that enrich our concept image and improve our intuition. I shall argue that this form of intuition need not conflict with the rigour required of a mathematical theory. In planning the curriculum I shall argue for a cognitive approach that takes account of the current conceptual imagery of the learner, and show that this is different from the logical approach seemingly dictated by the logic of the mathematics. To provide goals for the learner where the "advance organisers" of Ausubel prove inappropriate I shall argue for "generic organisers" on the computer, analogous to the concrete materials of Dienes and others, and show how these may be used in a simpler mode of operation, according to the theory of Skemp, to provide the conceptual foundation for more abstract concepts.

**The theory of Piaget**

Piaget was prolific in his writing on psychological development, and his work colours all the main advances in developmental psychology in recent times. But what many perceive as his main contribution, his theory of stages from sensori-motor through preconceptual, concrete and formal levels of development, seems to have little bearing on the learning of mathematics at higher levels. Indeed Ausubel et al. [1968] (page 230) criticize this global aspect of his theory:

> ... because of Piaget's tendency to underestimate the abstract thinking of young children and because such a high percentage of American high school and college students fail to reach this abstract level of cognitive logical operations.

Though Piaget estimates the beginnings of formal operational thought in children of eleven or twelve, Ausubel et al. [1968] note (page 238):

> Representative studies have indicated that only 15.6% of junior-high-school students ... 13.2% of high-school students ... and 22% of college students were at this level.

The concrete/formal distinction has proved to be a useful starting point in developing local hierarchies of difficulty in extensive studies such as Hart (ed.) [1981] in the 12-16 age range though these stop short of the calculus. Orton [1980a/b] suggested a similar local hierarchy in his analysis of calculus concepts, indicating that more formal aspects of the theory have a higher cognitive demand.

But it is characteristic of Piaget's original theory that he asserted that the movement from one stage to another could not be greatly accelerated by the effects of teaching. Differences of cognitive demand have often been used in a *negative* sense to describe student difficulties, rarely to provide *positive* criteria for designing new approaches to the subject. Papert [1980] summed it up:

> The Piaget of the stage theory is essentially conservative, almost reactionary, in emphasising what children cannot do. I strive to uncover a more revolutionary Piaget, one whose epistemological ideas might expand the known bounds of the human mind.

## Transition and mental reconstruction

A valuable aspect of Piaget's theory is the process of *transition* from one stage to another. Thus the learner goes through a period of cognitive conflict in which he must reorganize his ideas into a new state of mental equilibrium. Piaget used the term *assimilation* for the process by which the individual takes in new data. He called the changes in cognitive structure brought about by this process *accommodation*. Skemp [1979] distinguishes between the case where the process causes a simple *expansion* of the individual's ideas and the case where there is cognitive conflict, requiring *reconstruction*. It is the process of reconstruction which provokes the difficulties that occur during a transition phase, especially if reconstruction is inadequate and fails to resolve internal cognitive conflict.

## The Meaningful Learning Theory of Ausubel et al.

Although there are many aspects of the work of Ausubel et al. [1968] that differ significantly from that of Piaget, there are certain factors that are essential to the approach I shall advocate in this thesis.

The main thrust of Ausubel's work is the notion of meaningful verbal learning in the classroom through the student's active reception of ideas presented in an expository style by a teacher. He argues that, at best, it is only possible to design a *potentially* meaningful learning programme which takes account of what the learners already appear to know. He asserts that students can *meaningfully* learn concepts by actively linking the new ideas to "anchoring ideas" in their cognitive structure which will be modified in the process. He contrasts this meaningful form of learning with *rote* learning and distinguishes between the reception/discovery dimension in which new ideas are encountered and the meaningful/rote dimension in which the material is assimilated. Later in this chapter I shall advocate a pragmatic attitude to the reception/discovery dimension which attempts to maximise meaningful learning.

Fundamental to Ausubel's overall approach to meaningful learning is the notion of an *advance organiser*, which is

"Introductory material presented in advance of, and at a higher level of generality, inclusiveness, and abstraction than the learning task itself, and explicitly related both to existing ideas in cognitive structure and to the learning task itself ... i.e. bridging the gap between what the learner already knows and what he need to know to learn the material more expeditiously."

Such a principle requires that the learner has the appropriate higher level cognitive structure available to him. In situations where this may be missing, a different kind of organising principle will be necessary.

## The theory of Dienes

Dienes [1960] made a most valuable contribution to mathematics education through his structured materials for young children in the concrete operational phase and the supporting educational theory. He cites four major principles (page 30):

(1) The dynamic principle (in which the learner progresses through a series of stages in grasping a concept),

(2) The constructivity principle (in which concepts may be constructed intuitively before analysis),

(3) Perceptual variability (using different concrete

exemplars of a concept to abstract the concept itself)

(4) Mathematical variability (varying all the mathematical factors in a concept to allow it to be fully comprehended).

(The interpretations are mine, not his.)

Though these are intended for young children using concrete apparatus, extrapolating them to the use of the microcomputer in learning mathematics furnishes an organising principle that gives valuable learning experiences for more advanced students. For example, it will be possible for the individual student to explore mathematical concepts at an elementary level and gain a sense of the ideas before they are consolidated into formal theory. I shall explore the analogy between the concrete materials of Dienes, Gattegno and others designed for younger children and computer software that may be used by people of all ages to build and abstract concepts.

## The theory of Skemp

Skemp [1976,1979] distinguishes between *instrumental understanding* (recognising a task as one of a particular class for which one already knows a rule) and *relational understanding* (relating a task to an appropriate schema). This distinction is in accord with Ausubel's discussion of the meaningful/rote dimension.

Skemp [1979] proposes a learning theory in which intelligent learning is seen as a goal-directed activity. He postulates a delta-one system as a teachable director system operating on the physical environment and a delta-two system which operates on delta-one. Reflective intelligence has consciousness centred in delta-two and the objects of consciousness are concepts, schemas, plans or activities in delta-one. As the individual grows older and more sophisticated, it is advantageous for him to achieve voluntary control over his learning processes.

Skemp indicates how the emotions may be effected by moving towards or away from goals (which one wishes to achieve) and anti-goals (which one tries to avoid). In particular he suggests that the ability to move to a goal is signalled by confidence and inability by frustration, whereas the ability to move away from an anti-goal is signalled by security and an inability by anxiety.

He makes a valuable distinction between different modes of building and testing conceptual structures in the following table from Skemp [1979] (page 163):

REALITY CONSTRUCTION

| REALITY BUILDING | REALITY TESTING |
| --- | --- |
| *Mode 1* | *Mode 1* |
| from our own encounters with actuality: | against expectation of events in actuality |
| *experience* | *experiment* |
| *Mode 2* | *Mode 2* |
| from the realities of others: | comparison with the realities of others: |
| *communication* | *discussion* |
| *Mode 3* | *Mode 3* |
| from within, by formation of higher-order concepts by extrapolation,and beliefs: imagination, intuition: | comparison with one's own existing knowledge |
| *creativity* | *internal consistency* |

He claims (page 164) that:

> Pure mathematics relies heavily on mode 3, in varying degrees on mode 2 and not at all on mode 1.

By this he means pure mathematics *in its finished form*. In the learning process, Skemp is in full agreement with the work of Gattegno, Dienes, Montessori and many other mathematics educators through the the use of concrete materials for young children to work in mode 1.

At the higher levels of mathematical education, mode 1 activities are usually absent. It is my intention to use computer programs to provide a mode 1 environment for older students to explore mathematical concepts as a foundation for the more usual theoretical modes 2 and 3.

## Cognitive Existence

Mathematical concepts built in modes 2 and 3 may not seem to have the same status as those in mode 1 which can be manipulated physically by hand or on the computer screen. But a concept constructed mentally in mode 3 may be manipulated mentally and has a reality within the mind which I shall term *cognitive existence*. The reality of such a concept is given greater credence if others seem to share the reality and discuss the concept in a mutually meaningful manner in mode 2.

Thus negative integers have a cognitive existence once they have been mentally constructed and manipulated. This reality is one that most mathematicians would share. But there are a number of beliefs in the calculus that are not universal.

For example, the paper Tall [1980a] showed how infinitesimals could be imagined on a number line and, using formal mathematical definitions following Keisler [1976], it demonsrated how these could be manipulated in a metaphorical manner using "microscopes" and "telescopes". For me an infinitesimal has a cognitive existence which is shared by mathematicians who have studied non-standard analysis, or who have read the paper and have made the ideas their own. But there are good mathematicians that do not share this reality and deny the existence of infinitesimals. As mature mathematicians we may have sufficient confidence to agree to differ. But students who develop cognitive notions related to limiting processes that do not accord with the accepted shared mathematical schemas may not be so fortunate. The conflict between their mode 3 construction and the accepted notion in mode 2 may lead to great cognitive conflict. Placing examples of the concept in mode 1 externalises the notion and allows it to be discussed in a manner which does not threaten the self-image as it might if the concepts were purely internalised in mode 3 for discussion in mode 2.

## Corporate knowledge and individual growth

Formal theories are developed over the centuries by the activity of many experts working in mode 3, sharing ideas with other experts in mode 2 to produce corporate knowledge that may be communicated and preserved in written form. Modern mathematical theories created in this way are usually based on carefully chosen axioms and definitions, and employ a process of deductive reasoning to lead to highly complex theoretical constructs.

But the growth of concepts in a single individual does not follow such a sophisticated and logical path. Formal definitions and formal deductions are totally inadequate starting points for an unsophisticated learner lacking the cognitive structure to make sense of them.

Ausubel et al [1968] make the fundamental distinction between the *logical* content of concepts (determined by the formal theory) and the *psychological* content (idiosyncratic to the individual).

In this thesis the distinction will be made between two domains of discourse: the *cognitive domain*, which is concerned with the cognitive processes of individuals, and the *knowledge domain*, which refers to the corporate knowledge mutually agreed by cognitive processes of experts. A knowledge domain contains conceptual structures which have been processed in a highly refined way by many able individuals to produced a polished and organised theory, the cognitive domain refers to the manner in which learners of all abilities come to grasp the theory and individual experts proceed to use it.

## Concept Image and Concept Definition

Internal conflicts within concepts play an important role with students' cognitive difficulties. After a four year period of investigating these difficulties I came across the terms "concept image" and "concept definition" in a preprint of Vinner and Hershkowitz [1980]. Here the terms were used to describe two boxes in a diagram that represented the way an individual might refer mentally to one or the other. The words were exactly what I needed to describe the conflicts I had observed in the calculus over the previous four years and together we wrote the article Tall & Vinner [1981]. This included a broadening of the meanings of the terms which I described in the introduction in the following manner:

> The human brain is not a purely logical entity. The complex manner in which it functions is often at variance with the logic of mathematics. It is not always pure logic that gives us insight, nor is it chance that causes us to make mistakes. To understand how these processes occur, both successfully and erroneously, we must formulate a distinction between the mathematical concepts as formally defined and the cognitive processes by which they are conceived.
>
> Many concepts which we use happily are not formally defined at all, we learn to recognise them by experience and usage

in appropriate contexts. Later these concepts may be refined in their meaning and interpreted with increasing subtlety with or without the luxury of a precise definition. Usually in this process the concept is given a symbol or name which enables it to be communicated and aids in its mental manipulation. But the total cognitive structure which colours the meaning of the concept is far greater than the evocation of a single symbol. It is more than any mental picture, be it pictorial, symbolic or otherwise. During the mental processes of recalling and manipulating a concept, many associated processes are called into play, consciously and unconsciously affecting the meaning and usage.

We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures.

For instance the concept of subtraction is usually first met as a process involving positive whole numbers. At this stage children may observe that subtraction of a number always reduces the answer. For such a child this observation is part of his concept image and may cause problems later on should subtraction of negative numbers be encountered. For this reason all mental attributes associated with a concept, whether they be conscious or unconscious, should be included in the concept image: they may contain seeds of future conflict.

As the concept image develops, it need not be coherent at all times. The brain does not work that way. Sensory input excites certain neuronal pathways and inhibits others. In this way different stimuli can activate different parts of the concept image, developing them in a way which need not make a coherent whole.

We shall call the portion of the concept image which is activated at a particular time the *evoked concept image*. At different times, seemingly conflicting images may be evoked. Only when conflicting aspects are evoked *simultaneously* need there be any actual sense of conflict or confusion. ...

The definition of a concept (if it has one) is quite a different matter. We shall regard the *concept definition* to be a form of words used to specify that concept. It may be learnt by the individual in a rote fashion or more meaningfully be learnt and related to a greater or lesser degree to the concept as a whole. It may also be a personal reconstruction by the student of a definition. It is then the form of words that the student uses for his own explanation of his (evoked) concept image. Whether the concept definition is given to him, or constructed by himself, he may vary it from time to time. In this way a *personal* concept definition can differ from a *formal* concept definition, the latter being a concept definition which is accepted by the mathematical community at large.

For each individual a concept definition generates its own concept image (which might, in a flight of fancy, be called the "concept definition image"). This is part of the concept image. In some individuals it may be empty, or virtually non-existent. In others it may, or may not, be coherently related to other parts of the concept image. For instance the concept definition of a mathematical function might be taken to be "a relation between two sets A and B in which each element of A is related to precisely one element in B". But individuals who have studied functions may or may not remember the concept definition and the concept image may include other aspects, such as the idea that a function is given by a rule or a formula, or perhaps that several different formulae may be used on different parts of the domain A. There may be other notions, for instance the function may be thought of as an *action* which maps a in A to f(a) in B, or as a *graph*, or as a *table of values*. All or none of these aspects may be in an individual's concept image. But a teacher may give the formal definition and work with the general notion for a short while before spending long periods in which all examples are given by formulae. In such a case the concept image may develop into a more restricted notion, only involving formulae, whilst the concept definition is largely inactive in the cognitive structure. Initially the student in this position can operate quite happily with his restricted notion adequate in its restricted context. He may even have been taught to respond with the correct formal definition whilst having an inappropriate concept image. Later, when he meets functions defined in a broader context he may be unable to cope. Yet the teaching program itself has been responsible for this unhappy situation.

... the concept images of limit and continuity are quite likely to contain factors which conflict with the formal concept definition.

Some of these are subtle and may not even be consciously noted by the individual but they can cause confusion in dealing with the formal theory. The latter is concerned only with that part of the concept definition image which is generally mutually acknowledged by mathematicians at large. For instance, the verbal definition of a limit "$s_n \to s$" which says "we can make $s_n$ as close to s as we please, provided that we take n sufficiently large" induces in many individuals the notion that $s_n$ cannot be equal to s (see Schwarzenberger & Tall [1978]). In such an individual this notion is part of his concept definition image, but not acknowledged by mathematicians as part of the formal theory.

We shall call a part of the concept image or concept definition which may conflict with another part of the concept image or concept definition a *potential conflict factor*. Such factors need never be evoked in circumstances which cause actual cognitive conflict but if they are so evoked the factors concerned will then be called *cognitive conflict factors*. ... They only become cognitive conflict factors when evoked simultaneously.

In certain circumstances cognitive conflict factors may be evoked subconsciously with the conflict only manifesting itself by a vague sense of unease. We suggest that this is the underlying cause for such feelings in problem solving or research when the individual senses something wrong somewhere; it may be a considerable time later (if at all) that the reason for the conflict is consciously understood.

A more serious type of potential conflict factor is one in the concept image which is at variance not with another part of the concept image but with the formal concept definition itself. Such factors can seriously impede the learning of a formal theory, for they cannot become actual cognitive conflict factors unless the formal concept definition develops a concept image which can then yield a cognitive conflict. Students who have such a potential conflict factor in their concept image may be secure in their own interpretations of the notions concerned and simply regard the formal theory as inoperative and superfluous.

### Obstacles

The consideration of conflicts in thinking is widespread in the literature, for instance Papert [1980] (page 121) says:

> New knowledge often contradicts the old, and effective learning requires strategies to deal with such conflict. Sometimes the conflicting pieces of knowledge can be reconciled, sometimes one or the other must be abandoned, and sometimes the two can both be "kept around" if safely maintained in separate mental compartments.

In the previous chapter we saw how "obstacles" occurred in Cornu [1983]: they are pieces of knowledge useful in one context, which become part of the cognitive structure precisely because they are useful, but are later ill-adapted for new tasks.

In Skemp's terms an obstacle occurs when the mental schema is not properly reconstructed to take account of the new data. One may conjecture that, as with Skemp's theory of goals and anti-goals, cognitive factors involving conflict may cause emotional reactions and lead to inability to learn the formal theory.

## Intuition

The term "intuition" is used in many different ways. For instance, one may refer to the *process* of intuitive thinking or the *gestalt* produced by intuitive thought. The former is characterised by Bruner [1974] (page 99) in the essay "Towards a disciplined intuition":

> In virtually any field of intellectual endeavour one may distinguish two approaches usually asserted to be different. One is intuitive, the other analytic ... in general intuition is less rigorous with respect to proof, more oriented to the whole problem than to particular parts, less verbalized with respect to justification, and based upon a confidence to operate with insufficient data.

The latter is described by Fischbein et al. [1979] (page 5):

> We use the term intuition for direct, self-evident forms of knowledge.

Fischbein [1978] allows for different kinds of intuition (page 161):

> *Primary intuitions* refer to those cognitive beliefs which develop themselves in human beings, in a natural way, before and independently of systematic instruction.

> ... *Secondary intuitions* are those which are developed as a result of systematic intellectual training ... In the same meaning, Felix Klein (1898) used the term "refined intuition" and F. Severi wrote about "second degree intuition" (1951).

Secondary or refined intuitions are the product of an individual with a rich cognitive structure relating to a specific knowledge domain. Faced with a new problem in this context a solution springs immediately to mind. In [1980d] I characterized:

> ... the central property of intuition: the global amalgam of local mental processes using existing cognitive structure, as stimulated by a novel situation.

I can now express the same thing more simply:

> *Intuition is the brain's completion of incomplete sensory data.*

It is a natural function of the brain evolved over millions of years to interprete external stimuli from a small number of cues. The manner in which the brain completes the data depends on the nature of its cognitive structure. It is my contention that the purpose of education should be to create a rich cognitive structure that can respond in a positive and meaningful way to problems in the given knowledge domain, preferably with transfer to other areas as well. According to this view we should be looking for an approach to mathematics that provides powerful and appropriate intuitions.

**Brain function: parallel and sequential processing**

I have long been interested in how the brain functions in an attempt to gain insight into the processes of mathematical thinking and have published several simple papers that reflect this interest (Tall [1977b, 1978a/b]). Fundamental influences on my thought here have been the work of Hebb [1972] and the text of Young [1978]. Young sees the act of thinking as operating "programs of the brain", and acknowledges the necessity for meaningful links between ideas whilst expressing uncertainty as to how this occurs in the mind:

> Increasingly however it has been realized that storage depends greatly on meaning and relating new information to what Bartlett (1932), following Head (1926) called a schema or model. This is essentially the conception we are following, but the worry is that no one has been able to provide any clear notions about the neuronal organization that constitutes the building of a schema or program."
>
> (Young [1978] p.94.)

Oatley [1978] shows the limitations of brain research by analogy with what would happen if the research techniques were applied to a computer instead of a human brain. A fundamental factor is the nature of our own cognitive structure. We are only able to think with the cognitive structure that we have. We use the schemata in our minds to interpret events. Thus we speak of "programs of the brain" because our experience includes the programs on a computer, although we are at pains to underline that the brain program may be a different kind of phenomenon from the computer program.

In particular, brain activity includes features of parallel processing quite unlike the sequential processing of a computer. For example, the interpretation of visual sensations clearly involves different processes being carried out simultaneously.

Intuitive thinking, which relates global ideas together in a single gestalt, may also involve parallel processing in a manner which is antipathetic to the sequential processing required in logical deduction.

Some psychologists relate the different modes of thinking to different hemispheres of the brain. Glennon [1980] summarizes the findings "from many research studies" in the following form:

| <u>Left hemisphere</u> | <u>Right hemisphere</u> |
|---|---|
| Verbal | Visuospatial (including gestural communication) |

| | |
|---|---|
| Logical | Analogical, intuitive |
| Analytic | Synthetic |
| Linear | Gestalt, holist |
| Sequential | Simultaneous and multiple processing |
| Conceptual similarity | Structural similarity |

He prefaces this table with the caveat that "we who work in the instructional psychology of mathematics ... must not presume to be neurologists". That being said, if the table just given has any basis in fact then it will go a long way to explaining some of the most interesting phenomena in psychology.

Note that this too suggests a distinction between logical and intuitive thinking. This distinction causes difficulties in mathematics teaching because mathematicians are familiar with the logic and order of the knowledge domain of mathematics and see this order as a way to present the mathematics to the learner. Hart [1983] condemns the purely logical approach:

> We have all been brainwashed by the undeserved respect given to Greek-type sequential logic. Almost automatically curriculum builders and teachers try to devise logical methods of instruction, assuming logical planning, ordering, and presentation of content matter ... They may have trouble conceiving alternative approaches that do not go step-by-step down a linear progression ... It can be stated

> flatly, however, that the human brain is not organised or designed for linear, one-path thought. (page 52)

> ... there is no concept, no fact in education, more directly subtle than this: the brain is by nature's design, an amazingly subtle and sensitive *pattern-detecting* apparatus. (page 60)

> ... the brain was designed by evolution to deal with *natural complexity*, not neat "logical simplicities" ... (page 76)

He takes this antipathy to logic to extremes, arguing for "brain-compatible schools" that allow the learner to work in a way that reflects current knowledge of brain activity. For him learning becomes a fluid matter, with fixed classes replaced by temporary groupings "for only as long as the grouping shares a common purpose"; the progress of each student is to be monitored regularly and "attendance will be largely descheduled to reflect individual activities". It is a brave vision of the future and in some ways the flexibilities are being introduced into our schools. But the totality of the vision is as yet unrealistic in our culture.

Hart's vision, which concentrates on the individual *cognitive* domain to the exclusion of the shared *knowledge* domain is as extreme as a purely formal approach to mathematics which reverses the bias. Both aspects need to be addressed: the cognitive domain requires the learner to be given opportunities to develop a concept image that can support the more formal deductions in the knowledge domain. The rationality of the knowledge domain in mathematics is a powerful factor in enabling the learner to come to terms with the mathematical theory.

**The false opposition between intuition and rigour**

The terms "intuition" and "rigour" are regarded by many mathematicians as being exclusive so that an "intuitive" approach is one lacking essential mathematical rigour. In some sense there is a grain of truth in this distinction.

Intuition seems to involve parallel processing that is different from the step by step sequential processing required in rigorous deduction. Thus an intuition may arrive whole in the mind and it may be difficult to separate the components into a logical deductive order.

The matter is further complicated by the belief that intuitive ideas are often visual primary concepts so that an "intuitive approach" to calculus, say, is one through pictures based on a common belief which we all share.

Research discussed in the previous chapter shows that we do not all share these common intuitions. Students have a variety of concept imagery prior to the calculus. So there may be a mismatch between what a teacher might regard as geometrically "intuitive" and what is psychologically intuitive to the student.

A mathematician often takes a complex mathematical idea and "simplifies" it by breaking it into smaller components ready to teach each component in a logical form. From his viewpoint the components may be seen as part of a whole, but a student may just see separate pieces of a jigsaw puzzle for which he has never seen the total picture.

As an example, consider a mathematical analysis of the notion of a derivative $f'(x)$. This requires the notion of the limit of $(f(x+h)-f(x))/h$ as $h$ tends to zero, so mathematically the derivative must be preceded by a discussion of the meaning of a limit. To make the process mathematically easier, the limit process is initially carried out with $x$ fixed; only at a later stage is $x$ allowed to vary to give the derivative as a function. Thus the mathematical analysis suggests the sequence:

(1) notion of a limit,

(2) for fixed $x$, consider the limit of $(f(x+h)-f(x))/h$ as $h \to 0$,

(3) call this limit $f'(x)$ and allow $x$ to vary to give the derived function.

However, when the learner is at stage (1), the limit notion is mysterious to him because it seems "plucked out of the air" without any real reason. There are already cognitive obstacles here. At stage (2) the limit process has further obstacles detailed in the last chapter and in Tall and Vinner [1981]. As we shall see later in this thesis, the move from (2) to (3) is by no means as easy as it might appear; many students who come to the calculus using a standard approach are not able to "see" the derivative function $f'(x)$ as a graph, given the graph of the original function $f(x)$. Thus this last stage is not "intuitive" in a pictorial sense when approached via this mathematical sequence. Clearly it is necessary to reappraise the approach to the derivative in order to make it more cognitively intuitive. One method, which avoids breaking the concept into pieces that may cause the students cognitive difficulties, will be to use the computer to generate a gestalt for the concept as a whole. We shall return to this shortly.

First we should consider a theoretical resolution of the problem of the distinction between the mathematician's notion of intuition and that of the psychologist. One solution is to distinguish between the knowledge domain (as a shared mode 2 reality which has been refined and organised formally by communication between many individuals over many generations) and the individual cognitive domain (as conceived and interpreted in a single mind). This allows the shared knowledge domain to be organised in a formal development in a textbook, with chapters covering the structure of various parts of the theory in a logical arrangement.

By contrast, the individual cognitive domain within the mind of the learner grows steadily richer in structure, with the short-term aim of exploring relationships in the knowledge domain and a long term aim of coping with its global structure and logic. At no stage is the mind of the learner divided neatly into chapters and topic headings. But its structure can develop to yield powerful global intuitions which may then be checked by rigorous linear arguments. In this way cognitive intuition may complement and support mathematical rigour rather than the two modes of thought being in opposition.

## A cognitive approach

An approach to the curriculum which takes into account the learner's current cognitive state and structures the knowledge domain in a manner appropriate for learning I shall term a *cognitive* approach. This embodies a fundamental idea stated by Bruner [1974] (page 99):

> Obviously, the aim of a balanced schooling is to enable the child to proceed intuitively when necessary and to analyse when appropriate.

A cognitive approach must take into account the known obstacles that may occur and seek to resolve potential cognitive conflict in an appropriate manner. In Schwarzenberger and Tall [1978] we wrote

> ... the aim is to construct a schema which is conflict-free in the sense that there exist smooth paths linking one thought to another without the stress and instability introduced by oscillating from one concept to another.

I now see that such an aim is unrealistically optimistic. It is not even totally desirable. Cognitive conflict, suitably graded so that it is not too great to be threatening, provides a challenge to learning and a deep satisfaction when resolved. A curriculum designed to avoid conflict may be dull and uninspiring. Though we may wish to give each concept a coherent meaning, the idiosyncratic concept images of individuals and the persistence of cognitive obstacles make this unlikely. The human mind is too complex to be able to make its structure totally coherent.

The aim should be to provide the learner with potentially meaningful experiences to help develop concept imagery to give appropriate intuitions. This should be designed to satisfy immediate needs and the use of the ideas in practical applications, for many students may not continue to study the subject more formally. (In the case of the calculus, in the UK less than 10% go on to study mathematical analysis.) But there should also be a long term aim to develop conceptual imagery in which the analytic mode of thinking resonates with the rest of the cognitive structure instead of clashing with it. The ultimate goal of a cognitive approach to mathematics is a theory which is both *cognitively intuitive* and *mathematically rigorous* at one and the same time.

## Organisational Principles in a Cognitive Development

I earlier mentioned Ausubel's idea of an advance organiser (introductory material at a higher level of generality related to the task and the learner's cognitive structure). I have often used advance organisers in my own work. For example "The Foundations of Mathematics" (Stewart and Tall [1977b]) is intended to help the student grow from school mathematics to more abstract thinking processes and is in four sections. Each section is prefaced by an advance organiser describing what is to happen using analogy with an idea familiar to the student and asking the student to adopt a new attitude in thinking.

In "Complex Analysis" (Stewart and Tall [1983]) it is assumed that the student already has the theory of real analysis at his command. Here the advance organiser is outlined in the preface:

> Students faced with a course on 'Complex Analysis' often find it to be just that - complex. In the sense of 'complicated'.

> It is true, of course, that the proofs of some of the major theorems can demand a certain technical versatility. But in many ways, on a conceptual level, complex analysis is actually *easier* than real analysis; it just isn't always taught that way. ...
>
> To exhibit this inherent simplicity of complex analysis we have organized the material around two basic principles: (1) generalize from the real case; (2) when that reveals new phenomena, use the rich geometry of the plane to understand them. Our aim throughout is to encourage geometric thinking, with the proviso that it must be backed up by analytic rigour.

To form a bridge from the geometry to the analysis the text includes a theorem called the "path lemma". This translates informal intuitive ideas into sound analytic reasoning. Thus the book builds on the student's cognitive structure but ends with a formal theory consistent within itself.

An advance organiser, by definition, demands that the learner has appropriate general structures in his mind which he can use to set the new learning task in perspective. What happens when breaking new ground where there are no obvious general principles at his command? The notion of a limit in the calculus is one such new area where cognitive obstacles arise in the mind of the learner. It is a new area and requires a different technique.

~~The degree to which the curriculum should be taught or discovered is a matter of dispute amongst psychologists. Papert [1980] speaks of Piagetian learning as "learning without being taught" (page 7) or "learning without a curriculum" (page 31) and extols the "powerful ideas" that can grow out of a child's interaction with a microcomputer using the computer language LOGO.~~

**Generic Organisers**

The technique that I propose is to build up a new and more abstract concept using examples after the manner of Dienes. This involves the use of a "microworld" which Papert [1980] describes as (page 117):

> a self contained world in which certain questions are relevant and certain questions are not.

For instance, Papert describes a microworld for learning Newtonian mechanics (page 121) and asserts (page 122) that computer technology can be used to produce a new learning path through

> a computer-based interactive learning environment where the pre-requisites are built into the system and where learners can become active, constructing architects of their own learning.

I define a *generic organiser* to be a microworld which enables the learner to manipulate examples of a specific mathematical concept or a related system of concepts. The term "generic" means that the learner's attention is directed at certain aspects of the examples which embody the more abstract concept. Concrete examples of generic organisers include Cuisenaire rods and Dienes blocks. Examples on the computer include the microworld INTEGERS (Thompson [1984], Dreyfus & Thompson [1985]) which is a LOGO world for exploring the positive and negative integers, and EUREKA (ITMA [1984]) which enables the user to explore a simulation of the changing volume of water in a bath and relate it to a graphical representation.

My plan is to design computer programs that act as generic organisers. These provide dynamic graphical representations of mathematical concepts under the control of the user. They do not draw the general concept, only a *particular example* of that concept in action. But if the learner has his attention drawn to the specific generic attributes of that example, then this will provide potentially meaningful information to help in the abstraction of the general concept.

Note that this abstraction is a *dynamic* process. Attributes of the concept are first seen *in a single exemplar*. The concept itself is slowly built up by further expansion and refinement of ideas looking at a succession of exemplars. Sometimes a desired attribute is seen repeated in many examples and reinforced. On other occasions an example is considered which fails to have a property that is not part of the desired general concept. Care must be taken that all examples do not have a non-generic attribute which the learner may take subconsciously into his concept image to form a later obstacle to comprehension. If used appropriately, the image will build up in the mind of the user not just a static concept but a dynamic *process* featuring the concept in action.

Thus abstraction of a concept is not seen as a *set-theoretic* notion selecting properties in common from a number of examples as is sometimes suggested in mathematics educational theory. This wrong-headed notion comes from educators having the concept image of set-theory in their cognitive structure and wrongly using it to interpret psychological data! Abstraction is a dynamic *psychological* process refining and expanding the concept image in the mind of the learner.

### The merits of discovered and received knowledge

Papert [1980] extols the virtues of what he terms "Piagetian learning", which is "learning without being taught" (page 7), or "learning without a curriculum". He describes how "powerful ideas" can grow out of a child's interaction with a microcomputer using the computer language LOGO.

However, Leron [1985], whilst remaining enthusiastic about this vision of learning reports a fundamental tension between "Piagetian learning" and "powerful ideas". He suggests that, without appropriate guidance, the pupils may not find the "powerful ideas" and suggests the need for a study guide to complement exploratory activities.

The distinction should be made between a LOGO environment in which children are learning how to explore, construct and improve, and a knowledge domain where there are established principles which a student needs to understand to come to terms with a given theory.

Bruner, through his essay "The Act of Discovery" has often been regarded as one of the fathers of discovery learning. But in [1974] (page 14) he wrote

> ... I had some years before published a paper entitled "The Act of Discovery" ... which had been interpreted as the basis of a "school of pedagogy" by a certain number of educators. As so frequently happens, the concept of discovery, originally formulated to highlight the importance of self-direction and intentionality, had become detached from its context and made into an end in itself. Discovery was being treated by some educators as if it were valuable in and of itself, no matter what it was a discovery of or in whose service.

His essay "Some elements of discovery" [1974] page 84 redresses the balance, continuing to emphasise the value of the learner exploring and discovering ideas in suitable parts of the curriculum but with an initial word of caution:

> It seems to me highly unlikely ... given the centrality of culture in man's adaptation to his environment ... that biologically speaking one would expect each organism to rediscover the totality of its culture. ... It seems equally unlikely, given the nature of man's dependency as a creature, that this long period of dependency characteristic of our species was designed entirely for the most inefficient technique possible for regaining what has been gathered over a long period of time, that is, discovery.

As he puts it succinctly as one of the points in his "Patterns of Growth" (Bruner [1966] page 4):

> Intellectual development depends on a systematic and contingent interaction between a tutor and a learner, the tutor already being equipped with a wide range of previously invented techniques that he teaches the child.

Ausubel's distinction between the discovery/reception dimension and the rote/meaningful dimension suggests that meaningful learning can be achieved by active participation of the learner during the reception of knowledge. Unaided discovery learning by itself may fail or, worse still, it may lead to cognitive obstacles of the kind discussed in the last chapter: the learner may develop successful limited techniques within a restricted microworld but these may fail to adapt to new situations.

## Generic Organisational Systems

The use of generic organisers can only be *potentially* meaningful, in the sense of Ausubel. They cannot guarantee that the user will abstract the general concept. In fact, if it is used for unaided discovery the organiser might be misused and non-generic noise embodied in the implementation may distract the user or, worse still, lead to cognitive obstacles. Thus the learner usually requires an external *organising agent* in the shape of guidance from the teacher, a text book, or some other agency to point towards the salient generic features and away from misleading factors.

A *generic organisational system* consists of a generic organiser and an organising agent. With older students the guidance may initially be total, with the teacher demonstrating the ideas, but then students will need to explore and discuss the organiser to get a feeling for its potentialities. They may use it for structured discovery learning or structured practice. Using such a system will still lead to the formation of idiosyncratic concept images. Students may need further time for free exploration and discussion to iron out the creases in their understanding. Here the formation of obstacles that are potential conflict factors in future learning is still a distinct possibility. Therefore the organising agent should review the progress of the learner and provide feed-back to maintain direction. Though a book or computer assisted instruction system may do this to a limited extent, the kind of computer programs written using current technology appear inadequate to deal with the variety of problems and obstacles that may occur. Future computer systems, using vast memories and powerful new organisational principles such as those hinted at in the language PROLOG, may eventually prove more successful. But at the moment, and for some time in the future, in the organisation of human learning there is no substitute for the human teacher.

## Uses of a generic organiser in learning

The Cockcroft report [1982] in its now famous paragraph 243 take a pragmatic view of learning and teaching, suggesting a variety of approaches:

> Mathematics teaching at all levels should include
> opportunities for
> exposition by the teacher;
> discussion between teacher and pupils and between pupils themselves;
> appropriate practical work;

consolidation and practice of fundamental skills and routines;

problem-solving, including the application of mathematics to everyday situations;

investigational work.

Thus it is valuable for the teacher to introduce mathematical concepts but, as each student has a different cognitive structure and develops his (or her) own idiosyncratic concept image, it is necessary to allow him (or her) to enrich this image through exploration and discussion, to produce a coherent and stable conceptualization.

A good generic organiser should offer a focal point for all these activities:

(1) it provides an environment for mode 1 building and testing to support the development of formal theory in modes 2,3

(2) in demonstration the teacher may use it to focus attention intrinsically on the concept as embodied in the examples in the organiser and away from extrinsic motivation;

(3) discussion may be promoted between the teacher and students;

(4) used collaboratively, it may promote discussion between

students themselves;

(5) concepts may be explored intuitively in generic form before they are analysed;

(6) students may use it to enrich their conceptual imagery to give a gestalt for the total concept.

(7) investigations may be carried out individually or in groups;

(8) it may be used for problem-solving.

## Misuses of Generic Organisers: the Generic Extension Principle

If the generic organiser is used in an environment that is not properly controlled, then the student may abstract properties from the examples studied that are not part of the concept being modelled. As the human mind is a powerful pattern-detecting apparatus, patterns may be found that are not intended to be abstracted. For example, the functions that are typed into the Graphic Calculus computer programs are all combinations of standard functions and, with the exception of ABS, INT and SGN, these tend to be continuous and differentiable everywhere that they are defined. My experience is that students do not draw examples of graphs with "corners" if they are left to their own devices. Thus exploration without guidance may easily lead either to the belief that all functions are differentiable, or that every function is differentiable except at a finite number of exceptional points. Indeed the latter was the commonly accepted belief in the mathematical community in earlier centuries.

The gradient program draws the chord through two points on the graph with abscissae $x$, $x+c$ and, as $x$ varies, it usually gives a good picture of the gradient function for $c=1/1000$. The unguided user without seeing a counterexample might believe that it is always possible to obtain a good picture of the gradient of any graph using $c=1/1000$.

These are examples of a general principle which seems to occur in many contexts in mathematics, the *generic extension principle*:

> If an individual works in a restricted microworld in which all the examples considered have a certain property, then, in the absence of counterexamples, the mind assumes the known properties to be implicit in other contexts.

For example, children who have experienced whole number arithmetic believe addition and multiplication makes bigger, giving obstacles when the number system is extended to fractions or integers. Likewise older students familiar with certain finite experiences may believe that infinite concepts behave in the same way. Thus if all the objects in a limiting process have a certain property, the limit object has that property. If $a_n=1/n$, then the terms $a_n$ decrease but are never zero, so the "limiting object" is considered as a tiny non-zero number. Likewise, because 0.999...9 is always less than 1 for a finite number of places, 0.999...(recurring) is considered less than one. Or if a graph is in the form of a staircase, then successive half-size treads are inserted into the staircase to give a new one with half-size steps, the repeated process is thought to give a staircase with infinitesimally small steps rather than the mathematical limit, which is a straight line.

When we review the history of the subject in the next chapter we shall see many instances of the generic extension principle in action. This phenomenon occurs so often in human thinking that it is important to have it in mind when using generic organisers.

## Long-term strategy

If a generic organiser is properly designed and the organising agent acts effectively, the intuitive grasp of ideas offered by the organiser can provide a firm basis for the later development of the formal theory. This may depend heavily on the action of the organising agent attempting to ensure that the non-generic properties of the organiser do not act as distractors and form obstacles. Nevertheless, a well-designed generic organiser should contain the *potential* for insight into the later theory.

For example, the organiser I designed for differentiation shows that a differentiable function looks "locally straight". This leads naturally into the concept of local linear approximation and (years later) to the fundamental idea that differentiable manifolds are "locally flat". Likewise the example of the blancmange function which is nowhere locally straight leads on to a natural proof of the existence of an everywhere continuous nowhere differentiable function (Tall [1982a]). This example exhibits a weakness of the original organiser. By adding a tiny multiple of the blancmange function $b(x)$ to any differentiable function $f(x)$ one gets a new function $f(x)+kb(x)$ (where $k$ is very small). This is nowhere differentiable yet its graph on a computer screen looks indistinguishable from $f(x)$ which *is* differentiable. Thus the initial idea of looking along a curve to visualize its gradient is theoretically unsatisfactory: two graphs can look the same in a particular picture where one is differentiable and the other is not. The organisational system contains the seeds of the eventual replacement of the generic organiser. It leads to a higher plane where one realizes the need for a more rigorous theoretical formulation. At a simple level one realises that even if the gradient of the graph of sinx looks like cosx, a proof by formal manipulation is necessary. At a higher level the organiser can be used to lead to the formal definitions which give the much desired theoretical formulation in the knowledge domain, supported by a rich intuitive infrastructure in the cognitive domain.

Thus an organiser acts in a Piagetian manner, first within an environment where equilibrium is possible, then a dissonant property may be encountered that causes conflict and requires mental reconstruction to move into a new and richer equilibrium state.

The function of a good long-term generic organiser is first to be directly relevant to the current cognitive state of the learner, yet to contain the seeds of more subtle ideas that lead into later formal theory, should that prove necessary.

## Mathematical theory building

The *theory* of formal pure mathematics is clearly based on Skemp's modes 2 and 3. It is traditionally taught using the reality building of mode 2 with a lecturer or teacher describing the theory and tested by the individual in mode 3 with occasional opportunities of discussion with other students or in tutorial mode 3. There are formal theories in mathematics which claim to be logical entities in themselves, with no reference to actuality. Formal set theory is one such area. But this theory cannot be studied by an individual who does not already have a cognitive structure rich in ideas from naive set theory. I would suggest that the process by which mathematical theories become formalised is

(1) building up a coherent collection of linked ideas and developing chains of deduction linking them;
(2) selecting ideas at the beginnings of such formal chains to act as generative ideas or axioms for the theory.

(3) Checking that the major properties of the system can be deduced from these axioms;

(4) formalising the theory and detaching it from its intuitive beginnings.

These processes often take place over a period of centuries and it is only in the last hundred years that many areas of pure mathematics have been axiomatised.

As an individual comes to terms with this rich heritage, the role of a generic organiser is to help build the coherent collection of linked ideas which form the basis of the mathematical theory. If the organiser is later seen by the user to have theoretical deficiencies then these may be turned to good account to lead to the necessity for formalization and proof.

By using a generic organiser a student will be partaking in a mathematical process which he may make his own. It gives him a *goal* for his learning processes. This may be carried out in a way which encourages his exploration of the concepts and the formation of a relational understanding which can form the basis of a later logical understanding. Sensitively used as part of a long-term *cognitive* approach to the curriculum his conceptual imagery may grow from an intuitive grasp of the ideas to full operational mathematical thinking, both creative and analytic.

Bruner [1966] ends his search "towards a theory of instruction" with the comment:

> A body of knowledge enshrined in a university faculty and embodied in a series of authoritative volumes is the result of much prior intellectual activity. To instruct someone in these disciplines is not a matter of getting him to commit results to mind. Rather, it is to teach him to participate in the process that makes possible the establishment of knowledge. We teach a subject not to produce little living libraries on that subject, but rather to get a student to think mathematically for himself, to consider matters as an historian does, to take part in the knowledge getting. Knowing is a process, not a product.

# 4.Cultural and Historical Background

In this chapter those aspects of the historical development of the Calculus will be considered that contribute to today's cultural beliefs in the subject. No attempt will be made to write a full history; a perceptive overview was written nearly half a century ago by Boyer [1939] and still remains relevant, despite the fact that it was written before infinitesimal methods had been given a sound logical basis. First we shall consider ideas from the theories of cultural evolution (Wilder [1968]) and scientific paradigms (Kuhn [1962]) which give a framework for viewing historical development. Then we will review aspects of the history of the calculus which reveal the conflicts and differences of opinion which have characterised virtually every stage of its development.

Modern mathematical theory will not be used to pass judgement on which of two opposing historical views is more correct. Twenty years ago, before the development of non-standard analysis, we might be forgiven for seeing the history of analysis as the faltering gropings and giant leaps of previous generations towards the formal analytic theory that exists today. From such a viewpoint one might approve of a rigorous approach using limits and deny an approach using infinitesimals. The appearance of non-standard analysis made such a judgement untenable. By formulating a rigorous theory that used infinitesimals Robinson [1966] provided an alternative touchstone which showed the infinitesimal method in a new light. He even re-evaluated the theory of Leibniz from his new vantage point [1966] and showed how this theory was the fore-runner of his own. In Tall [1979a] I demonstrated a flaw in Robinson's view: the calculus of Leibniz is an essentially simpler theory than that of Robinson and has extra properties that are not found in the later theory. (Essentially, each infinitesimal in Leibniz's theory has a specific order which is a positive integer; this property does not hold in Robinson's non-standard analysis.) The moral of this story is that *no* modern theory may be used to form an absolute judgement of the achievements of the past.

## The cultural evolution of mathematics

Wilder's classic text [1968] takes an anthropological view of the historical development of mathematics. He sees the profound effect of cultural forces on the growth of mathematical ideas.

> Mathematicians themselves seem prone to ignore or to forget the cultural nature of their work and to become imbued with the feeling that the concepts with which they deal possess a "reality" outside the cultural milieu - in a sort of Platonic world of ideals. Indeed, some mathematicians seem to be completely lacking in the insight that the modern physicist has attained - the recognition that even his observations, as well as his concepts, are coloured by the observer. How much more this must be the case in

> mathematics, where the conceptual has gradually gained primacy over the observable?
>
> ... Attempts to change the direction of mathematical research by individuals .. seem to be of little avail. Only strong environmental and internal pressures, such as are sometimes imposed by war, dislocation forced by political changes, radical alterations in the host culture, "crises" in mathematics itself, and the like, appear to be effective in changing the course of mathematical development. (preface, pages viii,ix.)

Wilder refers to various cultural forces that operate when new ideas are introduced. Cultural elements move from one culture to another by a process of *diffusion*. New elements take time to become part of the culture even if they eventually succeed. He describes this *cultural lag* as a form of conservatism and cites the delay in introducing the metric system into the U.S.A., even though it became an integrated part of the scientific subculture.

Cultural lag has a cognitive basis that is more fundamental than an explicit conservative attitude. If an individual's cognitive structure is accustomed to working in certain patterns, there is no easy way of replacing those patterns wholesale. I can remember embracing the metric system and learning to "think metric" until I needed to extend a book-shelf that I had made in imperial measures several years before. The whole corpus of experience locked in my mind from the time when I built the shelf was suddenly re-awakened and, for a while, I viewed the problem in feet and inches, unable to think metrically at all. It is not always a conscious decision that causes conservatism; a far more potent force is the structure of the human mind that resonates with familiar patterns of thought and finds reconstruction a difficult process.

A more overt obstacle to the incorporation of new cultural elements is *cultural resistance*. Here it is the case that new elements are positively resisted rather than simply taking time to be absorbed. An example is the English use of the pint and the mile, despite the incorporation of other metric measures. In parts of Europe there are still pre-Napoleonic measures available such as the old "Pfund" in Germany (around 500 gms). In these cases the new elements did not have features that were obviously more useful to the culture than the old elements, so they failed to be adopted.

## The paradigms of Kuhn

Kuhn [1962] looks at various stages of scientific development throughout history when there is a relatively stable period of "normal science":

> Aristotle's *Physica*, Ptolemy's *Algamest*, Newton's *Principia* & *Opticks*, Franklin's *Electricity, Lavoisier's Chemistry* and

> Lyell's *Geology* - these and many other works served for a time implicitly to define the legitimate problems and methods of a research field for succeeding generations of practitioners. They were able to do so because they shared two essential characteristics. Their achievement was sufficiently unprecedented to attract an enduring group of adherents away from competing modes of scientific activity. Simultaneously, it was sufficiently open-ended to leave all sorts of problems for the redefined group of practitioners to resolve. Achievements that share these two characteristics I shall henceforth refer to as "paradigms" ... (page 10.).

As we review the history of the calculus we shall see that few periods are stable in this paradigmatic sense. Virtually all have inner conflicts, even competing theories pulling in different directions. Commenting on this phenomenon Kuhn asserts (page 199):

> If two men disagree, for example, about the relative fruitfulness of their theories, or if they agree about that but disagree about the relative importance of fruitfulness and, say, scope in reaching a choice, neither can be convicted of a mistake. Nor is either being unscientific. There is no neutral algorithm for theory-choice, no systematic decision procedure which, properly applied, must lead each individual in the group to the same decision. In

this sense it is the community of specialists rather than its individual members that that makes the effective decision.

## The Ancient Greeks

The Pythagorean School in the Fifth Century B.C. were known for their belief that (whole) numbers could be made the foundation of geometry. According to Boyer [1939] page 20,

> This hypostatization of number had led the Pythagoreans to regard the line as made up of an integral number of units.

When they found that there is no finite line segment so small that the diagonal and the side of square could both be expressed in terms of it, they were forced to question their view of the nature of matter:

> ... may there not be a monad or unit of such a nature that an indefinite number of them will be required for the diagonal and for the side of a square?

The *Method* of Archimedes, relates that Democritus was the first Greek mathematician to determine the volumes of the pyramid and the cone in the fifth century B.C. Though we do not know the method by which this was done, we do know that Democritus belonged to the Abderitic school which (according to Boyer [1939]):

> ... held that everything, even mind and soul, is made up of atoms moving about in the void, these atoms being hard indivisible particles, qualitatively alike but of countless shapes and sizes, all too small to be perceived by sense impressions. (page 21.)

According to Plutarch, Democritus identified a fundamental dilemma in his theory (Heath [1921], page 180):

> If a cone were cut by a plane parallel to the base, what must we think of the surfaces forming the sections? Are they equal or unequal? For if they are unequal they wil make the cone irregular as having many indentations, like steps, and unevennesses; but if they are equal, the sections will be equal and the cone will appear to have the property of the cylinder, and to be made up of equal, not unequal, circles: which is very absurd.

The Eleatic school proclaimed the apparent contradictions which arose either from assuming infinite divisibility of a line or from assuming that the line was made up of indivisible atoms or monads. The obvious objection to the Pythagorean monad, proposed by Zeno, was that if it has any length, an infinite number will constitute a line of infinite length, and if it has no length then an infinite number would have no length. Zeno's four famous paradoxes, come in two pairs: The "dichotomy" and "Achilles and the Tortoise" refute the idea of infinite divisibility, whilst the "arrow" and the "stade" refute the possibility of indivisible atoms of time and space.

Plato (c.428 – c.348 B.C.) discussed these basic difficulties in his *Dialogues*, including the Pythagorean problem of the nature of number and its relation with geometry, the difficulty of incommensurability, the paradoxes of Zeno and the Democritean dilemma.

Aristotle (384–322 B.C.) proposed the distinction between the actual infinite and the potential infinite, though he held different views for number and geometric magnitude (Boyer [1939], page 41):

> ... in the direction of largeness it is always possible to think of a larger number... Hence this infinite is potential, ... and not a permanent actuality, but consists in a process of coming to be, like time... With magnitudes the contrary holds. What is continuous is divided ad infinitum, but there is no infinite in the direction of increase. For the size which it can potentially be, it can actually be.

To describe the continuum, Aristotle suggested:

> By continuous I mean that which is divisible into divisibles that are infinitely divisible.

As they grappled with the idea of the continuum, the Greeks did not move towards a definition of irrational number. Instead Eudoxus developed the method of exhaustion to determine the comparative areas of two figures. This proved very successful in calculating the precise areas of a number of curved figures, but it had a basic flaw. It was not a general method that could be applied to a wide class of areas, each one needed a special argument to inscribe and circumscribe figures whose area could be calculated, in such a way that the outer area and inner area could be made successively closer. The area was not calculated by a modern limiting process, rather by a method of false position, showing by contradiction that it could be neither smaller nor larger than the desired result, using both the inner and outer areas in the process.

The *Method* of Archimedes revealed another dichotomy in Greek thinking. It used infinitesimal ideas to calculate the area of a parabolic segment, by regarding it as an aggregate of parallel lines. The published version of Archimedes' calculation of the area of this figure used the formal method of exhaustion. Over two millenia ago the double standards of analysis had begun. One might use an intuitive method to obtain an answer, but then it was necessary to formulate a rigorous proof to publish it in a form acceptable to the mathematical community. These different ways of thinking are quite typical of the separate processes of *building* a theory and *testing* it. But a false picture is left for later generations when the methods of theory-building are suppressed.

## The re-awakening

After the fall of Greece, the cultural forces required for the philosophical study of mathematics were no longer present. During the Dark Ages in Europe, the greatest minds turned to religious philosophy and it is largely the Moslem and Hindu civilizations that maintained mathematical culture.

During mediaeval times there are occasional sparks which show small developments in the calculus, such as the idea of motion under constant acceleration studied by Oresme (c.1360). Nicholas of Cusa (1401–1464) regarded the circle as a polygon with an infinite number of sides which inspired Kepler (1571–1630) to formulate a metaphysical "bridge of continuity" in which normal and limiting forms of a figure are categorized under one definition. Thus conic sections were seen as constituting a single family of curves (Kepler, *Opera Omnia II*, page 595), and there is no essential difference between a polygon and a circle, between an ellipse and a circle, between the finite and the infinite, and between an infinitesimal area and a line.

In 1586 Simon Stevin threw off the yoke of double reductio ad absurdum used in formal proof by exhaustion when he showed that the centre of gravity of a triangle lies on the median. His argument (later published in his *Hypomnemata Mathematica* of 1605) used inscribed parallelograms only, with bases parallel to the base of the triangle and centres of mass on the median. By increasing the number of parallelograms the difference between the triangle and the parallelograms could be made less then any given quantity. He actually calculated the ratio and showed that it could be made as close to the required value as desired. Perhaps it was his practical background as an engineer that took him to a more arithmetic view of the limiting process, but it was a view which was not shared by many of his contemporaries, who preferred to use pure geometry.

In 1609 Kepler showed that the area of a circle was Cr/2 where C was the circumference and r the radius, by regarding the circle as a regular polygon with an infinite number of sides and adding up the areas of an infinite number of infinitesimal triangles with base on the circle and vertex at the centre. By a similar method he saw the sphere composed of an infinite number of infinitesimal cones with vertices at the centre, whose bases made up the surface area of the sphere; he thus showed the volume of the sphere to be one third the product of the radius and the surface area. 1612 was a vintage year for wine and Kepler produced improved methods for calculating the volume of wine barrels using his new methods.

This work heralded a period in which areas and volumes were calculated by all sorts of *ad hoc* methods based on infinitesimals or indivisibles. In 1635 Cavallieri's *Geometria Indivisibilibus* used indivisibles to calculate areas and volumes. He theorized that a line is made up of an indefinite number of points, a plane of an indefinite number of lines, and a solid of an indefinite number of planes. He avoided speculation as to the nature of the infinite, neither sharing Aristotle's view of infinity as only a potentiality nor giving it a metaphysical significance after the style of Nicholas of Cusa and Kepler.

He obtained the area under the curve (in modern notation) $y=x^n$ from $x=0$ to a as $a^{n+1}/(n+1)$ by a process of summation, first for $n=2$ and later for $n=3,4,\ldots,9$, using his own geometric equivalent of the binomial theorem. He did this by adding indivisible lines. As Struik notes ([1969] page 218):

> Cavalieri's summation of lines into areas and of areas into volumes can easily trip the unwary, as we have observed. Cavalieri was well aware of it, but expected that the dificulties would be removed in due time, that to cut the Gordian knot could be left to some later Alexander, as he put it.

The introduction of coordinate geometry by Descartes (1637), anticipated in unpublished work of Fermat earlier in the decade, led to solutions of problems concerning tangents and areas under curves. For instance Fermat evaluated the area under the curve $y=x^{p/q}$ between 0 and x by using rectangular strips with basepoints at $x^{p/q}$, $(ex)^{p/q}$, $(e^2x)^{p/q}$... where $0<e<1$. He summed the areas using a geometric progression, substituted $e=E^q$ to obtain a quotient with $1-E$ as a factor on the top and bottom, cancelled this common factor, and set $E=1$ to obtain the area in the form $(p/(p+q))x^{(p+q)/q}$.

He also found maxima and minima of polynomials by equating $f(x)$ and $f(x+E)$. He realized that this was not an exact equality, but divided the "pseudo-equality" by E, then put $E=0$ to find the extrema.

Descartes used an ingenious method of determining tangents which, in modern notation, amounts to finding a circle through $(x,f(x))$ which intersects the curve in coincident points and taking the tangent to the curve to be the tangent to the circle. Roberval calculated the tangent to the parabola (a locus described by a moving point equidistant from a fixed point and a line) using a method akin to a parallelogram of velocities.

In England John Wallis read an account of Cavallieri's work, and sought to remove all traces of geometry from the method, assuming that any plane figure could be made up of an infinite number of straight lines or infinitesimally thin parallelograms. These were so thin as to be "scarcely anything but a line", yet not too thin so that "by an infinite multiplication a certain altitude or width can be acquired". In his books *De Sectiones Conicis* and *Arithmetica Infinitorum*, both published in 1655, he introduced the symbol ∞ and asserted that any rule that worked for finite numbers would also work for infinite numbers. Though the second book considered ratios that "at length differ by less than any assignable magnitude", he viewed an area as a totality of an infinite number of infinitely small parts rather than a limit.

The philosopher Thomas Hobbes preferred the geometrical approach of the Greeks and attacked Wallis's work bitterly, describing Arithmetica infinitorum as a "scurvy book" and the arithmetic as "a scab of symbols". Isaac Barrow, Newton's professor at Cambridge also preferred a geometrical approach and used the idea of instantaneous velocity to describe motion:

> To every instant of time, or indefinitely small particle of time, (I say instant or indefinite particle, for it makes no difference whether we suppose time to be made up of instants or indefinitely minute timelets); to every instant of time, I say, there corresponds some degree of velocity, which the moving body is considered to possess at the instant. (*Geometrical Lectures* 1670).

He viewed a figure as being made up of lines and admitted that very narrow rectangles should be substituted for the lines, maintaining "it comes to the same thing whichever way you take it".

His method of finding tangents was reminiscent of the method of Fermat, but using geometrical infinitesimals. The object of computation of tangents was not its gradient (which is the modern derivative) but the *subtangent*, which is the segment cut off on the x-axis between the intersection of the tangent and the foot of the perpendicular dropped from the point on the curve. Using these calculations, Barrow's *Lectiones Geometriae* contained a version of the fundamental theorem of the calculus that was *implicit* rather than explicit. (See Struik [1969] page 253.) Thus, when Newton came to put together his ideas on the calculus, the mathematical culture of the day was bursting with ideas ready to be organised.

## Newton

Newton began work on the calculus under Barrow in 1664 and his first manuscripts date from 1665, including his "pricked" letters $\dot{x}$, for which Leibniz later used the notation dx/dt. In *De Analysi per Aequationes Numero Terminorum Infinitas* (1669) he gave the bare essentials of his discoveries about infinite series and the binomial theorem, using these results to solve problems in finding areas and lengths of curves. Instead of calculating the area under a curve by summation, Newton used a new method by considering the momentary increase in the area at the point in question as an indefinitely thin rectangle. His method showed that, if the area under a curve is given by $z=(n/(m+n))ax^{(m+n)/n}$, then if the abscissa is increased from x to x+o (using the notation of Gregory), the augmented area is

$$z+oy = (n/(m+n))a(x+o)^{(m+n)/n}.$$

Expanding the latter by the binomial theorem, subtracting z and dividing by o, then neglecting any terms containing o, Newton found

$$y=ax^{m/n}.$$

He applied these same ideas to other curves. For the first time a powerful general technique for calculating areas became available, encompassing within it the "Fundamental Theorem of the Calculus" that exhibits differentiation and integration as inverse operations.

Newton was dissatisfied with his approach and had two more attempts at describing the processes. In *Methodus Fluxionem et Serium Infinitorum* (c.1671) he regarded a line to be generated by the continuous motion of a point, publishing the idea in 1704 as an appendix to his *Opticks*. An English translation by J. Stewart appeared in 1745 (see Struik [1969] page 303):

> I consider mathematical quantities in this place not as consisting of very small parts; but as described by continuous motion. Lines are described, and thereby

generated not by the apposition of parts, but by the continued motion of points.... I sought a method of determining quantities from the velocities of the motions or increments, with which they are generated; and calling these velocities of the motions or increments, *fluxions* and the generated quantities *fluents*. ... Fluxions are very nearly as the augments of the fluents generated in equal but very small particles of time, and, to speak accurately, they are in the *first ratio* of the nascent augments ...

Thus the fluent x after a very small time o becomes $x+\dot{x}o$ where $\dot{x}$ is the fluxion (neglecting all but the "first ratio").

His third method of approaching the theory was to use *prime* and *ultimate* ratios which he devised around 1676 and published in *Principia Mathematica* (1687). Here he distinguished between the "prime ratio" of finite magnitudes, say the ratio of $(x+o)^n-x^n$ to o, and the "ultimate ratio" which they approached, in this case $nx^{n-1}$ to 1.

Despite this long and tortuous mental struggle with the concepts, which clearly had a cognitive reality for him in a mode 3 sense, he was not able to convey them to others who did not share his reality in mode 2. Thus his work was ripe for criticism by such as Bishop Berkeley in the following century who took the position of the "plain man" and questioned the logic of his theory.

## Leibniz

In 1673 and 1676 Leibniz visited London and purchased a copy of Barrow's *Lectiones Geometriae*. His first published article on the calculus in the *Acta Eruditorum* appeared in 1684. This showed him with a strong concept image of the ideas of differentiation, though his expression of the ideas was such that others found his powerful vision ambiguous. His friends, the Bernouilli brothers referred to it as "an enigma rather than an explanation".

His first definition of the differential quotient dy/dx interpreted dx and dy as finite quantities, by taking dx to be a straight line selected arbitrarily, and the line, which is to dx as the ordinate is to the subtangent, is taken as dy. In modern terminology, the tangent is drawn to the curve and the components of the tangent vector are dx,dy.

Lebniz's problem was that this definition requires the a *priori* knowledge of the tangent. To use his theory to find the tangent, Leibniz then had to use infinitesimal ideas.

> We have only to keep in mind that to find a *tangent* means to draw a line that connects two points of the curve at an infinitely small distance, or the continued side of a polygon with an infinite number of angles, which for us takes the place of the curve. (Quoted from the translation in Struik [1969] page 276.)

To carry out his calculations, Leibniz's experience showed him that infinitesimals come in different orders. For instance,

$$(x+dx)^2=x^2+2xdx+dx^2$$

includes a quantity $dx^2$ which of the second order compared with the quantity $xdx$, which is of the first order. By performing calculations with infinitesimals and casting out higher order terms, he was able to formulate a general principle to find the formulae for derivatives.

At various times in his life his attitude towards the nature of these infinitesimals varied (Struik [1969] page 280):

> Leibniz (like Newton) was never very consistent in his explanation of differentials. For instance, in his reply to his critic, the Dutch physician Bernard Nieuwentijt (1654–1718), who rebuked him for rejecting infinitely small quantities as if they were nothing at all ... he answered that it was correct to consider quantities of which the difference is incomparably small to be equal; a line is not lengthened by adding a point. It was only a question of words, he added, whether one rejected such an equality. ... And later he pointed out; "There are different degrees of infinity or of infinitely small, just as the globe of the Earth is estimated as a point in proportion to the distance

> of the fixed stars, and a play ball is still a point as compared to the radius of the terrestial sphere, so that the distance of the fixed stars is an infinitely infine or infinite of the infinite with respect to the diameter of the ball".

In a letter to Grandi (*Mathematische Schriften IV*) he wrote

> ... we conceive the infinitely small not as a simple and absolute zero, but as a relative zero ... that is, as an evanescent quantity which yet retains the character of that which is disappearing.

At other times he referred to infinitesimals as "useful fictions", admitting that no-one could ever prove or disprove the existence of infinitely small quantities, but this was irrelevant since anything that could be done with infinitesimals could be done without them. Here he invoked a form of the law of continuity of Kepler and Cusa, which he expressed clearly in a letter to Bayle of January 1687:

> In any supposed transition, ending in any terminus, it is permissible to institute a general reasoning, in which the final terminus may also be included.

This idea, which recurs in the thinking of modern students in the phenomenon I have termed a "generic limit", shows Leibniz extrapolating his finite experiences to the infinite limiting case just as Wallis did with numbers. The law of continuity is clearly false for more complex examples, as the "monsters" of the nineteenth century show, with the (pointwise) limit of continuous functions on occasion failing to be continuous. But it must be remembered that the functions of Leibniz's experience were more restricted than those of later generations, and I have argued in Tall [1981a] that the weaknesses attributed to Leibniz's theory in modern times would not necessarily apply to the examples Leibniz would have encountered in his own mathematical culture.

Leibniz's theory of integration appeared in 1686 in a casual way, as part of a review of a book by John Craig, a Scottish pupil of Newton. Here he uses his integral notation $\int y\, dx$ as an infinite sum of infinitesimal elements and performs a calculation in which "sum and difference, or $\int$ and d, are each others' converse".

He never published his theories in text-book form. That task fell to L'Hopital, whose *Analyse des Infiniment Petits* was published in 1696.

## Criticism and new formulations

The eighteenth century was a time of consolidation of techniques and a quest for a universally acceptable foundation for the subject. The unfortunate dispute over who was the original inventor of the calculus, Newton or Leibniz, led to a division between the work in England and on the continent, with the English school using the Newtonian "pricked" variable notation and the continentals using the differentials of Leibniz.

Nieuwentijt criticized Leibniz's method and rejected the use of higher order infinitesimals. Berkeley fiercely attacked both Newton and Leibniz in his pamphlet *The Analyst* or *A discourse Addressed to an Infidel Mathematician Wherein it is examined whether the object, principles, and inferenced of the modern Analysis are more distinctly conceived, or more evidently deduced, than religious Mysteries and points of Faith.* Criticizing Newton's theory Berkeley writes (taken from Berkeley [1951] page 66 et seq.):

> The Method of Fluxions is the general key by help whereof the modern mathematicians unlock the secrets of Geometry, and consequently of Nature. ... Lines are supposed to be generated by the motion of points, planes by the motion of lines, and solids by the motion of planes. And whereas quantities generated in equal times are greater or lesser according to the greater or lesser velocity wherewith they increase and are generated, a method hath been found to determine quantities from the velocities of their generating motions. And such velocities are called fluxions.
> ... And of the aforesaid fluxions there be other fluxions, which fluxions of fluxions are called second fluxions. And

> the fluxions of these second fluxions are called third fluxions: and so on, fourth, fifth, sixth, &c. *ad infinitum*. Now, as our sense is strained and puzzled with the perception of objects extremely minute, even so the imagination, which faculty derives from sense, is very much strained and puzzled to frame clear ideas of the least particles of time, or the least increments generated therein... And it seems still more difficult to conceive the abstracted velocities of such nascent imperfect entitites. But the velocities of the velocities, the second, third, fourth and fifth velocities, &c., exceed, if I mistake not, all human understanding. The further the mind analyseth and pursueth these fugitive ideas the more it is lost and bewildered...

Equally scathing is his attack on Leibniz, L'Hopital and Nieuwentijt (page 67):

> The foreign mathematicians are supposed by some, even of our own, to proceed in a manner less accurate, perhaps, and geometrical, yet more intelligible. Instead of flowing quantities and their fluxions, they consider the variable finite quantities as increasing or diminishing by the continual addition or subduction of infinitely small quantitites. Instead of the velocities wherewith increments are generated, they consider the increments or decrements themselves, which they call differences, and which are

supposed to be infinitely small. The difference of a line is an infinitely little line; of a plain an infinitely little plain. They suppose finite quantities to consist of parts infinitely little, and curves to be polygons, whereof the sides are infinitely little, which by the angles they made one with another determine the curvity of the line. Now to conceive a quantity infinitely small, that is, infinitely less than any sensible or imaginable quantity, or than any the least finite magnitude is, I confess, above my capacity. But to conceive a part of such infinitely small quantity that shall be still infinitely less than it, and consequently though multiplied infinitely shall never equal the minutest finite quantity, is, I suspect, an infinite difficulty to any man whatsoever; and will be allowed such by those who candidily say what they think; provided they really think and reflect, and do not take things upon trust.

... All these points , I say, are supposed and believed by certain rigorous exactors of evidence in religion, men who pretend to believe no further than they can see. That men who have been conversant only about clear points should with difficulty admit obscure ones might not seem altogether unaccountable. But he who can digest a second or third fluxion, a second or third difference, need not, methinks, be squeamish about any point in divinity. There is a natural presumption that men's faculties are made alike. It is on this supposition that they attempt to argue and convince one another. What therefore shall apear evidently impossible and repugnant to one may be presumed the same to another. But with what appearance of reason shall any man presume to say that mysteries may not be objects of faith, at the same time that he admits such obscure mysteries to be the object of science?

In the years that followed there were various attempts to paper over the cracks perceived in the theory. The difficulty of viewing the integral as an infinite sum was avoided by many authors by considering integration simply to be the reverse process to differentiation. Bernoulli, Euler, L'Huilier and Lagrange were just a few of the authors who began the modern trend to view Leibniz's notation in the form $\int f(x)\, dx$ to mean "the indefinite integral of f(x) *with respect to x*", where "indefinite integral" now meant what we now term an "antiderivative" (a function whose derived function was f(x)). L'Huilier wrote a prize-winning essay for the Berlin Academy (published 1787) in which he retained the symbol dy/dx but insisted that it should be considered as a limiting quantity and not as a quotient of dy by dx.

In 1797 Lagrange's *Theorie des fonctions analytiques* proposed a new formulation based on power series

$$f(x+i) = f(x) + pi + qi^2 + ri^3 + \ldots$$

which he showed, by formal manipulation, to give

$$f(x+i) = f(x) + f'(x)i + f''(x)i^2/2 + f'''(x)i^3/(2.3) + \ldots$$

It was from this source that the expression "differential coefficient" came, as $f'(x)$ is the coefficient of $i$ in the expansion for $f(x+i)$.

Lagrange used this method, not only to determine the derivatives of elementary functions with numerous applications to geometry and mechanics, he also gave "proof" of the theorem that every continuous function could be expanded in this way. It should be noted that Lagrange's idea of continuity was different from the modern $\varepsilon-\delta$ definition. At this time a continuous function was one given by a single formula.

The limitations of this approach were exposed when Fourier showed that discontinuous curves could be represented by means of infinite series of differentiable functions at the beginning of the next century. Examples introduced in the 1820s by Cauchy, such as $e^{-1/x^2}$, which is infinitely differentiable, but does not have a power series expansion at the origin, effectively killed off the power series method in real analysis. Strangely, power series proved to be the motivating factor in complex analysis. (See Stewart & Tall [1983].)

In 1803 the Cambridge mathematician Woodhouse brought together the English and the continental schools of thought in his textbook *Principles of Analytical Calculation*. He showed the cumbersome nature of the "pricked" variable notation in higher derivatives, where the tenth derivative, say, would require ten dots above the variable, and proposed his own modification for the Leibniz notation. He used the notations $\delta x$ and $\delta y$ for corresponding increments in x and y, $\Delta x$ and $\Delta y$ to denote corresponding infinitesimal increments, and dx, dy to denote the differentials. Thus if $y=x^2$, one would have

$$\delta y = (x+\delta x)^2-x^2 = 2x\delta x+\delta x^2 \text{ (for finite } \delta x)$$

$$\Delta y = 2x\Delta x+\Delta x^2 \text{ (for infinitesimal } \Delta x)$$

and

$$dy = 2xdx \text{ (for infinitesimal } dx).$$

Woodhouse's notation is the origin of the modern $\delta$- and d-notation, though subsequently the meaning of the notation would change to come into line with L'Huilier and others, who dismissed infinitesimals and saw dy/dx as a limit, not as a quotient.

Cauchy contributed three great treatises on Differential Calculus, his first being *Cours d'analyse de l'Ecole Polytechnique* of 1821. Here he gave the definition of a continuous function (translated in Boyer [1939] page 277):

The function f(x) is continuous within given limits if between these limits an infinitely small increment i in the variable x produces always an infinitely small increment, f(x+i)−f(x), in the function itself.

Cauchy gave the notion of a limit in the modern $\varepsilon$-$\delta$ form, though the number line on which he used the definition still did not have the modern real number formulation and may be perceived to include infinitesimal quantities. He now *defined* an infinitesimal as a particular kind of variable:

> One says that a variable quantity becomes infinitely small when its numerical value decreases indefinitely in such a way as to converge to the limit zero. (quoted from Boyer [1939] page 273.)

Bolzano made great strides towards the modern notions of analysis, refining ideas earlier associated with L'Huilier (Boyer [1939] page 269):

> He defined the derivative of F(x) for any value x as the quantity F'(x) which the ratio $(F(x+\Delta x)-F(x))/\Delta x$ approaches indefinitely closely, or as closely as we please, as $\Delta x$ approaches zero, whether $\Delta x$ is positive or negative. ... Euler had explained dy/dx as a quotient of zeros. ... Bolzano, however, emphasized the fact that this was not to be interpreted as a ratio of dy to dx or as the quotient of zero divided by zero, but was rather one symbol for a single

> function. He held that a function has no determined value at a point if it reduces to 0/0 ... and he correctly indicated that, by adopting the limiting value as the meaning of 0/0, the function may be made continuous at this point.

He tended to publish his ideas in private pamphlets which were less widely read than those of Cauchy, so that his ideas sometimes went unnoticed. These included ideas on the completeness of the reals and the continuity of functions that pre-dated more well-known publications. In 1834 he gave an example of an everywhere continuous, nowhere differentiable function, predating another classic example of such a function given by Weierstrass in 1872.

## The rigorous formulation

In the remainder of the nineteenth century the arithmetization of analysis was carried out, through formal definitions of the real line by Dedekind cuts or Cauchy sequences, and formal definitions of limits and continuity using $\varepsilon$–$\delta$ methods in a purely arithmetic form by Weierstrass.

This led Boyer [1939] to claim:

> The unequivocal symbolism of Weierstrass may be regarded as effectively banishing from the calculus the persistent

notion of the fixed infinitesimal.

The name given to the axiom that distinguishes the real numbers in their rigorous formulation from the rational numbers: the *completeness axiom* suggests an air of finality, as if there is "no room" for any more numbers on the number line. Even today one sometimes hears a mathematician assert that there are "gaps" in the rational line that are "filled in" by the irrationals.

Then Cantor added a further blow with his infinite cardinals that could be added and multiplied but not subtracted or divided. If one cannot divide by an infinite cardinal, how can one have an infinitesimal?

Persistent infinitesimals

None of these blows completely removed infinitesimal ideas from the culture. A typical popular textbook on calculus, Abbott [1940], is still in print after 45 years, and discusses the limit of the function $(x^2-2)/(x-2)$ as x tends to 2 in the following terms (page 24):

> ... as the value of x approaches 2, the value of the fraction approaches 4, and that ultimately when the value of x differs from 2 by an infinitely small number, the value of the fraction also differs from 4 by an infinitely small number.

This cultural element still exists within our society and is being transmitted from generation to generation, despite the existence of a conflicting rigorous formulation which is shared by a large proportion of the current mathematical community. In writing a popular Encyclopaedia of Mathematics, West et al. [1982] describe calculus as:

> ... a branch of higher mathematics that deals with variable, or changing, quantities. Calculus is based on the concept of infinitesimals (exceedingly small quantities) and on the concept of limits (quantities that can be approached more and more closely but never reached).

## Non-Standard Analysis

The non-standard analysis of Robinson [1966] formulated infinitesimal calculus in a rigorous way. But it was phrased in a language of logical propositions, logical variables, filters and ultrafilters which were not the common property of most pure mathematicians. The latter had moved from a "variable" language for functions to a set-theoretic language. Non-standard analysis was not received with the acclaim that Robinson expected, as the resolution of a profound conflict that had been with us for centuries. In moments of private self-doubt he referred to it as his "step-child". Despite the attempts to introduce it as a method for teaching the calculus through the text of Keisler

[1976] and the "proof" of Sullivan [1976] that the material was better understood by the students, the peak of its use in this context is long past, with Keisler's text now out of print (1985).

Though mathematicians pay lip-service to the infinitesimals of non-standard analysis, most had too much effort invested in the standard theory for it to give way. There is even a meta-theorem that states that "any standard theorem in analysis that can be proved by non-standard means also has a standard proof". Thus the cultural elements in non-standard analysis do not produce any new standard theorems, and the new cultural elements are not seen to have positive advantages over the old, leading to a classic case of cultural resistance.

There are still researchers actively working in the subject forming sub-cultures of their own. Perhaps their work will eventually diffuse into the wider mathematical community as has happened with so many other new ideas in history. The day for open acceptance of infinitesimals may yet come again, but the predominant view of modern analysis keeps them currently out of centre stage.

## Cognitive aspects of historical development

Reviewing the history of the calculus, it becomes clear that virtually every age had its conflicts. In each case we see the mathematicians of the period working with concepts that had a cognitive existence for them in a culture where other experts shared their viewpoint. Yet there were usually aspects that produced conceptual difficulties that could not be explained within their culture and provoked differences of opinion.

Many of the dilemmas of previous ages can be "resolved" using subsequent theory. But this does not solve the original problem. It is rather like someone who uses numbers for counting purposes. Concepts such as "three objects" or "five objects" have a mode 1 reality. But "minus two objects" has no meaning in this context. The fact that one may see the numbers 3 and 5 on a number line, and extend that number line in the opposite direction to give a positional meaning to −2 may provide a wider context in which negative numbers have a meaning, but this does not resolve the original problem in its original context.

Zeno's comment, that an infinite number of monads of non-zero length would together form an infinite length, is seen to be false in a non-standard paradigm. If N is an infinite non-standard integer, then N monads may form a unit length if each monad has infinitesimal length 1/N. Such a response would be unlikely to find favour in the Greek culture.

Likewise it is clear from the writings of Newton and Leibniz that their experiences gave a notion of infinitesimals a cognitive reality which Berkeley was unable to share. Anyone who is willing to accept the superreals (Tall [1980a]) will have no problem in imagining infinitesimals of any order: $\varepsilon^2$ is of order 2 and $5\varepsilon^{31}+1066\varepsilon^{57}$ is of order 31. In Tall [1980a] these infinitesimals are elements of a non-Archimedean ordered field which may be visualized using formally defined "microscopes" with a geometrical interpretation. Such ideas are common property of a sub-culture of modern mathematicians but they would hardly satisfy Berkeley in terms of the culture in which he lived.

What mathematicians did in each culture was to get on with their practical activities, despite theoretical inconsistences that may occur. We still do this today in using axiomatic systems to form the basis of pure mathematical theories, despite the fact that Godel showed in 1931 that any formal system including the integers must contain theorems that are true, yet cannot be proved within the given theory.

The lessons of history

We would be wise to take note of the lessons of history. No individual can hope to influence the system unless it is through using ideas that are consonant with the culture and with the changing trends in the culture. Ideas which are novel will not be accepted unless there is a soil in the culture to nourish their growth. Changes in approach will only diffuse slowly into the system, and to be successful they must have aspects which show clear superiority over cultural elements already present. Even if this happens, the new ideas will coexist with useful ideas retained in the current culture.

Just as the experts in history gained a cognitive belief in mathematical concepts through using them, we may be able to help students gain an insight into ideas of the calculus by providing an environment in which they can explore and manipulate the ideas to give them a cognitive reality.

As we have seen in the theory of generic organisers, the computer may well provide a mode 1 context for student exploration and manipulation. The computer is also a possible agent to bring about a change in paradigm. We shall consider this possibility and propose a plan of attack in the chapters which follow.

## 5. Review of possible modern approaches to the calculus

Before embarking on the design of a cognitive approach to calculus, it is wise to consider the various alternative approaches currently available. A number of alternative approaches to differentiation considered in Tall [1981a] will be reviewed, together with other possibilities. The corresponding approaches to integration will be outlined. This exercise shows a wide variety of meanings for the concepts: a legacy of the cultural developments of history.

The arrival of the computer is drastically changing the scene. In the American College system topics from finite mathematics are competing with continuous calculus and challenging its accustomed dominance. New symbolic manipulators giving automatic symbolic differentiation and integration suggest that the long apprenticeship in techniques for differentiation and integration may soon be relegated in importance to the same level as long-division in the age of the calculator. At such a time which has all the hall-marks of a change in paradigm, one should consider carefully the place of a new cognitive approach to the calculus if it is intended to be widely applicable.

### Different views of the differentiation

In Tall [1981a], comments were made on the difficulty and validity of a number of possible approaches to the calculus, highlighting the varying meanings given to the same symbols in different theories. The approaches considered were as follows.

The *old, intuitive, infinitesimal method*, uses an infinitesimal increment dx to calculate (f(x+dx)−f(x))/dx and then neglects any infinitesimal quantities after performing the calculation.

The *dynamic limit method*, with h as a variable real number, calculates the ratio of (f(x+h)−f(x))/h and allows h to dynamically get closer to zero. If the quotient approaches a fixed limiting value, this value is taken to be f'(x). The alternative Woodhouse-style notation, using increments $\delta x$, $\delta y$ is also allowed, with the limit being denoted by dy/dx. Here dy/dx is usually considered to be a single indivisible symbol, though taking dx to be any (non-zero) real number allows dy to be defined as dy=f'(x)dx, to give dx,dy as the increments to the tangent (so that (dx,dy) is the direction of the tangent vector).

The *numerical method*, tabulates specific values of h against (f(x+h)−f(x))/h to see what happens numerically when h is small.

The *computer drawing method*, magnifies a small part of the graph of a function. If under high magnification the graph looks almost straight, then the gradient of this almost straight line is taken as the gradient of the graph.

The *epsilon-delta method*, uses formal definitions of limits.

The *modern infinitesimal method* of Robinson [1966] embeds the real numbers in a larger hyperreal number system including infinitesimals. Every finite hyperreal number $\alpha$ is uniquely of the form a+k where a is real and k is infinitesimal and the standard part of $\alpha$ is defined to be the real number a. The derivative of f(x) is defined to be the standard part of (f(x+dx)−f(x))/dx where dx is now an infinitesimal hyperreal number.

The article defends Leibniz's method in the context of his own culture, upholding his strong cognitive insight into the theory, but acknowledges that it is no longer appropriate in the broader context of modern analysis. In today's mathematical culture the epsilon-delta method and the modern infinitesimal method both lay claim to provide a sound basis for formal analysis and the difficulties of the two methods were compared.

> ... The dynamic limit method provides a natural introduction to epsilon-delta techniques and also provides basic intuitions for the modern infinitesimal method.
>
> The processes and proofs in modern infinitesimal calculus are easy because they mirror cognitive processes and proofs in the dynamic limit method. They are hard in the epsilon-delta approach because of the complicated computations and the many quantifiers required to formalize

> the dynamic limit process without resorting to infinitesimals.
>
> The rigorous concepts of modern infinitesimal calculus are hard because of the difficulty of setting up the ordered field structure of the hyperreal numbers. They are made worse when the approach demands a perceptive use of logical language and first order predicate calculus as a pre-requisite.
>
> ... The infinitesimal method sometimes promises more than it can deliver because its construction is based on the axiom of choice and is therefore ... non-constructive.

The article concluded with the comment:

> *... whatever the balance of future developments, a blend of the dynamic limit method, with practical numerical computations and high magnification drawings is likely to provided the most suitable grounding for beginning calculus students, to prepare them for future refinements.*

There are other approaches to the calculus that differ in some respects from those given above. For instance, the SIMS study (McLean et al [1984]) considered the use of a different notation, $D_x f$ for the derivative of f at x:

Teachers stressed the $f'(x)$ or $y'$ and $dy/dx$ notation but virtually all mentioned the alternative notation $D_xf$ as well. Teachers and students both used $dy/dx$ notation and felt that $D_xf$ notation provided students with the most difficulty. The most insight was provided through use of $dy/dx$ notation.

In Israel the modern syllabus designed for schools by Amitsur goes considerably further, approaching the derivative via the *best linear approximation* using the "little o" notation. Here a function $g(h)$ is said to be $o(h^n)$ if $g(h)/h^n$ tends to zero as $h$ tends to zero, and one seeks a linear map $D_xf$ (depending on $x$) so that

$$f(x+h)=f(x)+D_xf(h)+o(h) \qquad (*).$$

In the one-dimensional case the linear map $D_xf(h)$ is just a constant multiple $kh$ (where $k$ depends on $x$).

The intention of this approach is to provide a definition that generalises naturally to differential geometry in higher dimensions, so that if $f:D\to\mathbb{R}^n$ where $D\subset\mathbb{R}_m$, then the derivative at $x\in D$ is the linear map $D_xf:\mathbb{R}^m\to\mathbb{R}^n$ satisfying condition (*), now interpreted in $\mathbb{R}^n$. The chain rule here states that

$$D_x(g\circ f) = D_{f(x)}(g)\circ D_x(f).$$

This is a classic example of taking a more advanced definition in mathematics and reformulating it for beginning students. Its danger is that students may not have the sophistication required to come to terms with it in the initial stages.

In Britain, the Schools Mathematics Project approaches the derivative via the notion of "scale factor". The average scale factor for a function $y=f(x)$ over an interval from $x=a$ to $x=b$ is

$$(f(b)-f(a))/(b-a).$$

By representing the mapping $x \mapsto f(x)$ in a set theoretic manner from one vertical line to another, the average scale factor represents the change in scale. The "scale factor at a point" is calculated by taking a and b closer and closer together. The value of this representation is that it is particularly good for interpreting the composite of two functions. As the change in scale near a point is given by multiplying by the scale factor, clearly the composite of two functions changes scale by multiplying first by the first scale factor, then by the second. Thus the scale factor can be used to illustrate the chain rule. Apart from this, it is basically a variant of the dynamic limit method.

If the scale factor approach is generalized to higher dimensions, it gives the *determinant* of the linear map $D_x f$ and therefore only generalises to functions where the range and domain are in spaces of the same dimension.

Other variants are possible, through viewing the derivative $f'(a)$ as a measure of how fast the function values of $f$ are changing near $a$, or more specifically as the instantaneous speed of a moving object.

A most promising method, extending that mentioned in [Tall 1981a] is to use a *global* approach, moving an extended chord along the graph, plotting the chord-gradient in the process. This gives an approximation to the graph of the derivative at the outset and can be valuable for exploration and investigation.

Integration

The *old infinitesimal method* of Leibniz regards the integral $\int y\,dx$ as the sum of an infinite number of infinitesimal strips width $dx$. As an increment $dx$ gives an additional area $dA = y\,dx$, the derivative of the area is $dA/dx=y$.

In modern times the historical view of a line made up of indivisible points or a plane out of lines is no longer held. Instead an area is calculated by breaking it up into thin strips and making some kind of estimate of the height of each strip. Formally this leads to the Riemann integral. The fundamental theorem of integration then reveals the inverse relationship between integration and differentiation.

The *dynamic limit method* in the Woodhouse notation takes a sum $\Sigma f(x) \delta x$ which is interpreted to mean the sum of strips width $\delta x$ with height $f(x)$ (for some $x$ in the strip-interval). If the sum tends to a limit as the maximum width $\delta x$ gets small, then then this is denoted by $\int f(x)\, dx$, now interpreted as a single symbol and read as "the (indefinite) integral of $f(x)$ with respect to $x$". Often the indefinite integral is identified with the antiderivative (a function $F$ such that $F'=f$) and the area from a to b is denoted by the symbol $\int_a^b f(x)\, dx$.

It should be noted that the (dx,dy) interpretation from differentiation does not carry over in an obvious way to give a meaning to the dx symbol in the integration formula (though we shall see it proves to be of great value when we turn to differential equations).

A few simple areas, such as $\int x\, dx$, may be calculated as formal algebraic sums. The sum $\Sigma x \delta x$ with equal strip-widths can be calculated using the formula for the sum of the first k integers. The approach generalizes, but at great cost; to calculate $\int x^n\, dx$ requires the formula for the sum of the nth powers of the first k integers, which accounts for the effort Cavallieri required to prove all cases up to n=9 in 1635.

The *numerical method* allows the integral to be computed by partitioning the interval [a,b] into strips and estimating the area of each strip. This is most easily done by taking all the strips to be the same width (with the last strip possibly being different to make up) and using a specific rule to estimate the area. For example it could be approximated as a rectangle using the first, last, or middle ordinate in each strip, or as a trapezium, or later by a quadratic approximation as in Simpson's rule.

The numerical calculations can be carried out on an ordinary hand calculator and Neill and Shuard [1982] testify to the value of this approach as a class exercise. However, if the calculator is not programmable, the sheer effort of calculation is formidable. Simply calculating the approximate area under the graph of $y=x^2$ from 0 to 1 is arduous and may not lead to a clear idea of the *exact* value. Using a step-width 0.1 and calculating lower and upper sums (taking the lower and upper ordinates in each strip), Orton [1985] obtains the values 0.285 and 0.385. These are hardly conducive to a good guess for the area. Using the trapezium rule with the successive step-widths 0.2, 0.1 and 0.05 he obtains the values 0.340, 0.3340, 0.33375. This illustrates a possible limiting process and may even suggest a limit of 0.333... or 1/3, but it does not move us much closer to getting the area from 0 to x for arbitrary x.

A *computer drawing method* can be used in a variety of ways, the most obvious to show how the area is closely approximated by the strips when the strip-width gets small. But an insightful approach is to use all the information in the calculation and to plot the approximate area $A(x)$ from 0 to x as each strip is added on. For small strip-width this gives an idea of the *graph* of $A(x)$. It can be used to conjecture not just the *value* of the area in a single instance, but the *formula* for the global area function $A(x)$. Figure 5.1 shows the graph of $f(x)=x^2$ drawn over the ranges x=0 to 5, y=0 to 25 and the dots represent the value of the cumulative area under the graph calculated from zero using the mid-ordinate rule. The graphs cross where x=3, y=9. One may conjecture that the area graph is a higher power of x. If it is of the form $kx^3$, then substituting x=3 gives 27k=9, so k=1/3 and the area function may be postulated to be $A(x)=x^3/3$.

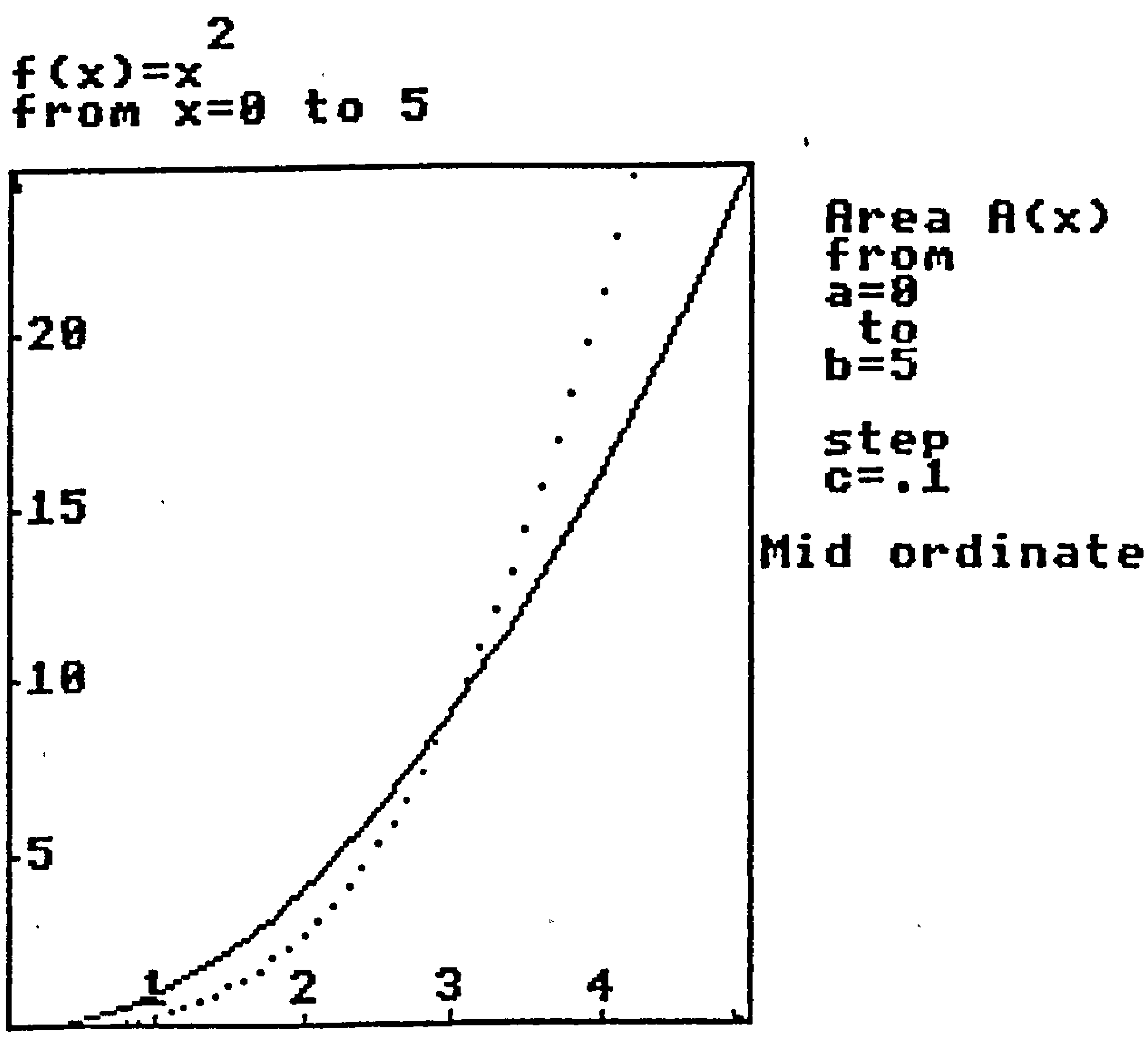



Figure 5.1

The *formal* $\varepsilon$-$\delta$ *method* uses the Riemann integral. The interval $[a,b]$ is partitioned, $a=x_0<x_1<\ldots<x_n=b$, a point $t_k$ is selected in each sub-interval $[x_{k-1},x_k]$ and the sum

$$\Sigma\, f(t_k)(x_k-x_{k-1})$$

is calculated. The limit is a specific constant K so that, given $\varepsilon>0$, a $\delta>0$ can be found such that all partitions with maximum strip-width less than $\delta$ give a sum within $\varepsilon$ of K.

Variants of this definition are often used, but it is clearly inappropriate for beginning calculus.

The *modern infinitesimal method* takes an infinite integer N, forms the infinite sum

$$\Sigma\, f(a+n\,dx)\,dx$$

from n=1 to N where $dx=(b-a)/N$, then defines the area as the standard part of this sum.

There are technical difficulties here for the student, detailed by Schwarzenberger [1978], which militate against using this approach in a beginning calculus course although it offers an alternative formal framework at a later stage.

Of the approaches considered, the most plausible initial approach to area again seems to be through dynamic limits and numerical calculations illustrated with computer drawings to give cognitive support. The antiderivative may be considered separately, either before or after the limit of a sum and the two separate ideas linked together in the fundamental theorem.

## The Computer

The computer is bringing new possibilities at a rapid pace, though the forces of cultural resistance are slowing down the diffusion process. When the Cockcroft Report was published in 1982, the computer had made little impact in the classroom. Microcomputers at that stage often had a rudimentary version of the BASIC language and usually lacked high resolution graphics. In the intervening three years change has been so fast that the BBC computer, hailed as the one of the most promising computers in Education in 1982, is now looking distinctly old-fashioned. Its memory is too small to cope with various new facilities, such as the symbolic differentiation procedures.

New features now available on computers relevant to the calculus include:

1. fast numerical algorithms,
2. dynamic high resolution graphical display,
3. symbolic algorithms for differentiation and integration,
4. increasingly powerful computer languages,
5. interactive software.

Not only will there be a variety of new approaches to the calculus using some of these facilities, the new computer paradigm is already creating applications which compete for the time given to calculus in the syllabus. Ralston [1984] is advocating an increase in the use of a wide variety of discrete mathematical applications which compete with the calculus.

In Tall [1985] I put the case:

> Computers are rapidly providing facilities for complex calculations and symbolic manipulations, but these often only give the *results* of the processes, without displaying the processes themselves. It will fall upon mathematics educators to provide ways of developing an understanding of underlying mathematical processes so that the results may be used with greater insight.

In Britain the Mathematical Association [1985] are advocating the use of short algorithms, written or modified by students, to gain insight in the mathematical processes. At the moment these programs are written in (BBC) BASIC and are usually purely numerical, though it is possible to add a graphical element without too much difficulty (see chapter 5 of the Mathematical Association's "132 Short programs" [1985]). As languages become more flexible and develop new facilities, this aspect of calculus is bound to increase in importance.

In Tall [1984a] I argued that the methods of discrete numerical analysis and continuous calculus are complementary. By utilising the numerical methods on a computer, illustrated with moving graphics, one could hope to gain a foundation for the practical ideas of the calculus that could provide the foundational support to capitalize on symbolic manipulators as well as leading on to the formal theory if desired.

Already various American researchers are exploring ways in which symbolic mathematical manipulation packages may be used for exploration of mathematical concepts (for example, Lane [1985], Stoutemyer [1985]) but these do not remove the need to understand the underlying mathematical processes.

There is still an important place for generic organisers on the computer to allow interactive exploration of the basic concepts of derivative, integral and differential equation. Such organisers will be all the more valuable if they plant suitable ideas in the cognitive structure on which to build a variety of later theories, including standard $\varepsilon$-$\delta$ analysis, non-standard analysis, the ideas of multi-dimensional differential geometry and the practical world of calculus to be used in applications.

# 6. A Cognitive Approach to the Calculus using Computer Graphics

In the previous chapter we considered a number of different possible ways of approaching the calculus. Chapter 3 suggests that an approach which provides an initial gestalt for each concept promises a distinct advantage in cognitive terms. But chapter 4 indicates that any change that is paradigmatic in the sense of Kuhn may be met by cultural resistance. It is not enough to produce "research evidence" to show the advantages of the new system. For example the work of [Cummins 1960] requires extra effort by the teacher to use an "experience discovery approach" producing more understanding but not significantly better performances in standard examinations. Such an approach is unlikely to overcome the natural inertia of teachers used to their own ways which prove equally successful under traditional evaluation.

Likewise the non-standard approach by Keisler has been resisted by the mathematical community at large, even though [Sullivan 1976] has shown that it has certain advantages. I have personal experience of cultural forces at work in this case. Having successfully taught such a course as an option for mathematics students at Warwick University, permission was refused to teach it again as a "mathematics option" though it would be permissible as an option from the education faculty. The Head of the Mathematics Department explained that a member of staff wished to give "History of Mathematics" and, if "Non-Standard Analysis" were also allowed, that would make two "non-mathematical" options available amongst the official mathematics courses. Non-standard analysis was not regarded as bona-fide pure mathematics.

To achieve change one must use the cultural forces currently available to one's advantage. In the case of a new approach to the calculus in Britain the influx of computers into school provided a need for appropriate software. There is a cultural pressure to use the computer with expected cultural resistance from those wishing to maintain the status quo.

The computer programs in Graphic Calculus were designed as generic organisers that could be used flexibly in a variety of ways and provide a basis for later developments in standard or non-standard analysis. Even so, their acceptance is by no means guaranteed, especially as there are elements in the approach (such as the idea of magnification) which are currently absent from the culture.

## Program design

The programs are intended to fit in with a level of understanding that is typical of students starting the subject. Currently computer programming is not part of the mathematics course and so the programs had to be immediately appealing to users unfamiliar with the computer. Thus it was decided that all input should be in normal mathematical notation. Although one could soon get used to writing the derivative of x↑n as n*x↑(n-1), it was easier to use cursor control up and down to type

$$x^{n} \text{ and } nx^{n-1}.$$

The design of a facility to translate "algebra" to BASIC took three months effort (including familiarization with 6502 machine code).

The programs are designed for ease of use without instruction beyond a brief review of facilities. This is achieved by using a "local menu" system that always has the options available cued on screen; any selected option takes the user through an obvious sequence of activities which requests the parameters required for a particular routine.

Every attempt is made to avoid using technical terms on screen that might not be understood by the user. Thus "Supergraph", which is a graph-drawing program used for pre-calculus experience, allows letters to be used as constants in an expression such as y=mx+c. But the menu does mention this facility. A less-sophisticated user might type such simpler expressions as y=3x+2; only when a formula involving letters is typed will the value of the constant be requested and the option to change the constant be added to the visible menu.

The routines are also designed to have variable speeds, so that they may be slowed or paused during teacher demonstration or speeded up when the user later becomes more expert.

It is essential that the programs be mathematically correct. Some computer programs currently available claim to draw "the derivative" when they actually draw a numerical approximation. Others claim to calculate the area by "upper" or "lower" sums when they have no universal routine to find the maximum or minimum values of a function to calculate these sums properly. As far as possible, such intellectually dishonest claims should be avoided.

All the graph-drawing routines should have error-trapping so that they can cope when the functions drawn become undefined; they should also draw vertical asymptotes. The programs should give satisfactory displays even when used in unusual circumstances that might occur during exploration or problem-solving.

Satisfying all these demands took over three years of program development and trials in schools.

## Generic organisers for the calculus

Three sets of programs have been designed, each with three programs which are generic organisers for specific mathematical concepts. The three programs for differentiation are "MAGNIFY" (to investigate the behaviour of graphs under high magnification), "GRADIENT" (to approach the gradient of a graph dynamically) and "BLANCMANGE" (to give an extreme example of a graph that is not differentiable).

The integration package has "ANTIDERIVATIVE" to approach antidifferentiation through the use of "direction diagrams" to draw a curve with given derivative, "AREA" to draw the areas under a graph using simple rectangle rules taking the sign of the area into account, and "SIMPSON" to extend the area drawing program to cover the Trapezium and Simpson rules.

The differential equation package has "1stODE" to extend the idea of antidifferentiation to an equation dy/dx=f(x,y) using a direction diagram, "2ndODE" to solve a second order differential equations in a similar way even though a direction diagram no longer seems appropriate and "3D-ODE" to draw three dimensional representations of simultaneous ordinary differential equations in which the direction diagram returns as the fundamental solution principle.

Each program is a generic organiser allowing flexible demonstration and investigation of specific mathematical concepts. For example the program GRADIENT is designed to give a gestalt for the gradient of a graph by looking along it and seeing the gradient vary. Seen after the program MAGNIFY it provides potentially meaningful insight into the gradient of a graph as a dynamic idea.

What follows is an outline of the overall approach to the calculus using these programs. It is a modified version of the paper [Tall 1985e]. Further details are found in the books accompanying the programs themselves [Tall 1985b/c/d].

## Differentiation

GRADIENT has two major subroutines available; one to demonstrate the numerical limit of the gradient of the chord tending to the gradient of the tangent and the other to draw the gradient function over an interval.

Traditionally the foundation of the calculus is the notion of a limit, either of a chord approaching a tangent or algebraically as a ratio $(f(x+h)-f(x))/h$ as h tends to zero. The computer brings a new possibility to the fore. Instead of viewing the idea of differentiation as calculating the gradient of the tangent to a graph, we may begin by considering *the gradient of the graph itself*. Although a graph may be curved, under high magnification a small part of the graph may magnify to look like a segment of a straight line. In such a case we may speak of the gradient of the graph as being the gradient of this magnified (approximately straight) portion. Figure 6.1 exhibits the graph of $y=x^2$ near $x=1$ approximating to a straight line segment of gradient 2.

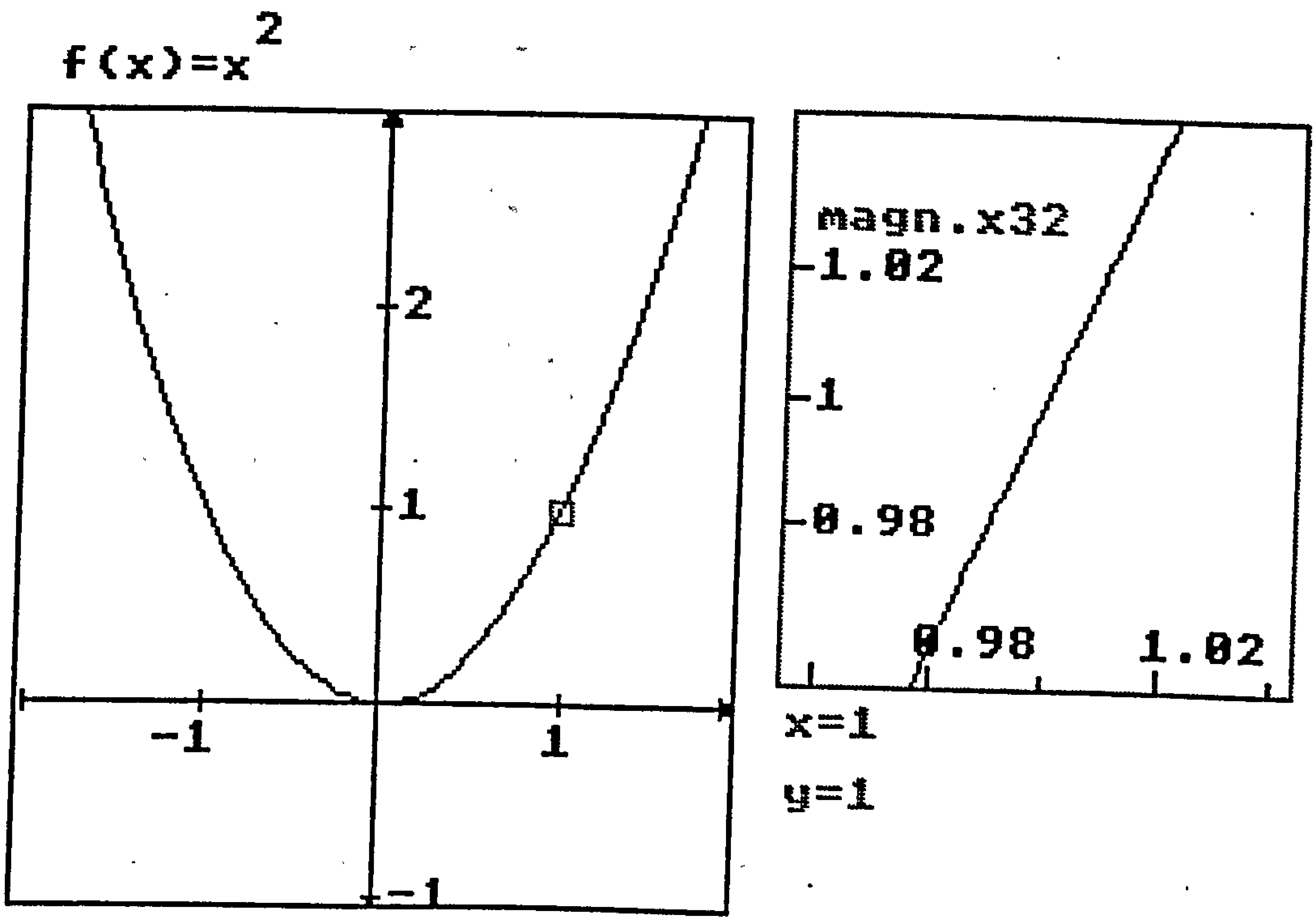



Figure 6.1

To represent the changing gradient of a graph, it is a simple matter to calculate the expression (f(x+h)-f(x))/h for fixed h as x varies. The program GRADIENT includes a routine that moves in steps along the graph drawing the chord through the points x, x+h on the graph and plotting the gradient of the chord as it proceeds. Figure 6.2 shows the gradient of the graph of sinx being built up. It clearly approximates to cosx; by superimposing the graph of cosx for comparison the gradient function may be investigated experimentally before any of the traditional formalities are introduced.

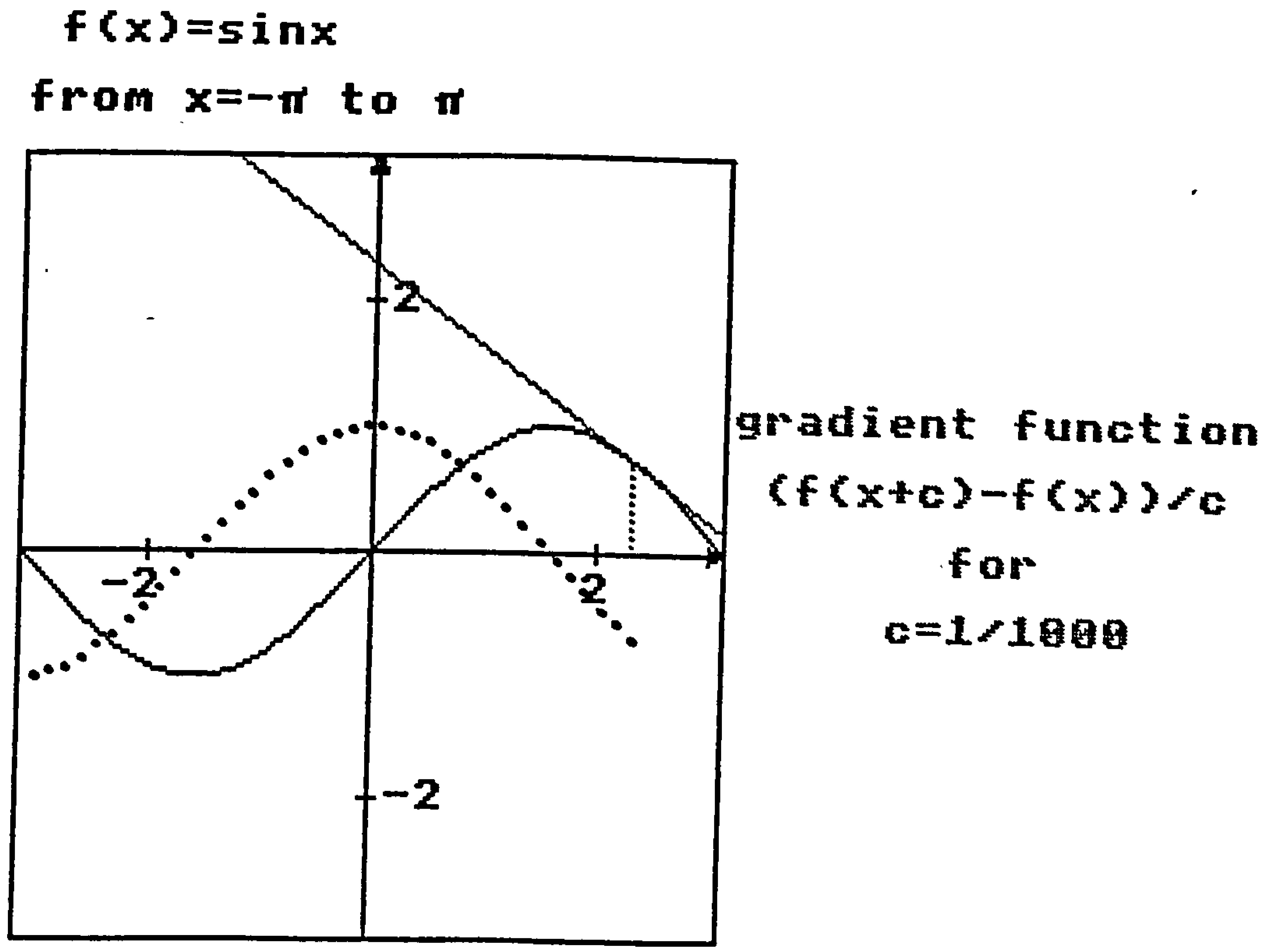



Figure 6.2

One product of this type of investigation is that it doesn't require a very small value of h to get a good computer picture of the gradient. It leads to the question: why bother to take limits at all? The answer is given by investigating a graph such as $f(x)=1/x$ which has a disconnected domain of definition. This graph clearly has negative gradient everywhere but, for any fixed value of h, there are values of x and x+h on either side of the origin whose chord has *positive* gradient. Thus the need to take limits arises from a purely practical consideration of handling functions whose domains are disconnected, to make sure that the gradient is only calculated between points on the same connected component. This leads naturally into the formal consideration of limits and the development of the formulae for calculus.

In developing the formulae, the symbols dx and dy can be given a meaning, dx being an increment in x and dy the corresponding increment in y, not to the graph, but to the *tangent* to the graph. Better still, one may view (dx,dy) as a *vector* representing the direction of the tangent, a valuable idea when we come to look at the meaning of differential equations.

There are other bonuses. The program naturally copes with positive or negative values of h in the formula $(f(x+h)-f(x))/h$ and pictures are often drawn with negative gradients instead of the traditional graph of an increasing function drawn in the majority of text-books. The moving graphics give a dynamic interpretation of the changing gradient, which in turn helps to expand the students' mental image of the concept.

"Intuitive" approaches to the calculus usually explain what a derivative is, without saying what it isn't. With the computer graphic approach it is easy to show what a non-differentiable function looks like. A function is not differentiable at a point if its graph near the point doesn't magnify to look straight. For instance a function may have different left and right derivatives, which simply means that the magnified graph looks like two straight half-lines meeting at an angle. The left and right derivatives at the point are the gradients of the corresponding lines. Simple examples include $|x^2-1|$ at $x=\pm1$ and $|\sin x|$ at multiples of $\pi$.

It is possible to draw a function everywhere continuous and nowhere differentiable (a concept previously beyond the reach of elementary calculus): its graph is so wrinkled that it never magnifies to look straight. The program BLANCMANGE draws such a graph and allows investigation of its properties. This leads to insights that may be converted into a formal proof that the function is nowhere differentiable, as in Tall [1982a].

## Integration

Integration involves two entirely separate concepts: anti-differentiation and summation processes such as finding the area under a graph.

Anti-differentiation is usually viewed as the reversal of the process of handling the formulae for differentiation and is largely seen by students as a problem-solving exercise in manipulating formulae. Graphically it may be characterized as knowing the gradient $dy/dx=f(x)$ of a graph and requiring to find a graph $y=I(x)$ fitting this information. The program ANTIDERIVATIVE draws short line segments through an array of points $(x,y)$ with the gradient $f(x)$. A solution $y=I(x)$ is simply traced out by following the direction of the lines (figure 6.3).

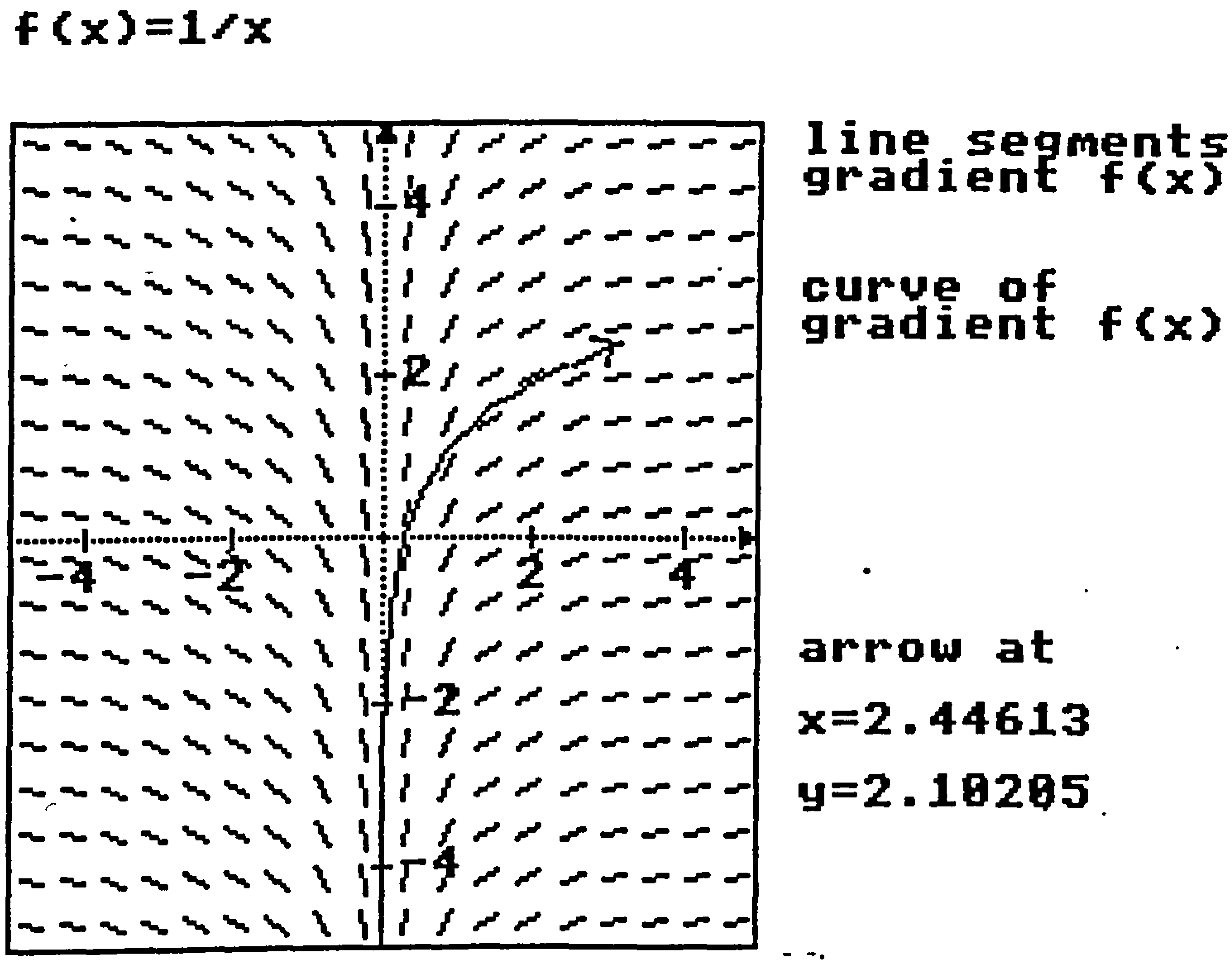



Figure 6.3

As the gradient direction is a function of x alone, the solution curves clearly differ by a constant. The program draws solutions numerically using a step along the graph rather than a fixed x-step. In doing so it remains on a connected component of a solution. When solving the equation dy/dx=1/x, a solution curve starting to the right of the origin always remains on the right. Thus the fact that two antiderivatives differ by a constant is seen only to be true over a *connected component* of the domain, a considerable advance on the limited view in most elementary courses where the antiderivative is given in the form I(x)+c for an "arbitrary constant" c, without mention of any restriction on the nature of the domain.

The program AREA calculates the area between the graph and the x-axis by a variety of methods. The numerical values are displayed and each part of the area is drawn using different colours for positive and negative results. Students can see that a positive step gives a positive result when the graph is above the axis and negative when below (figure 6.4). They can see equally well that a negative step reverses the signs, a concept traditionally regarded as difficult yet clearly represented by moving graphics.

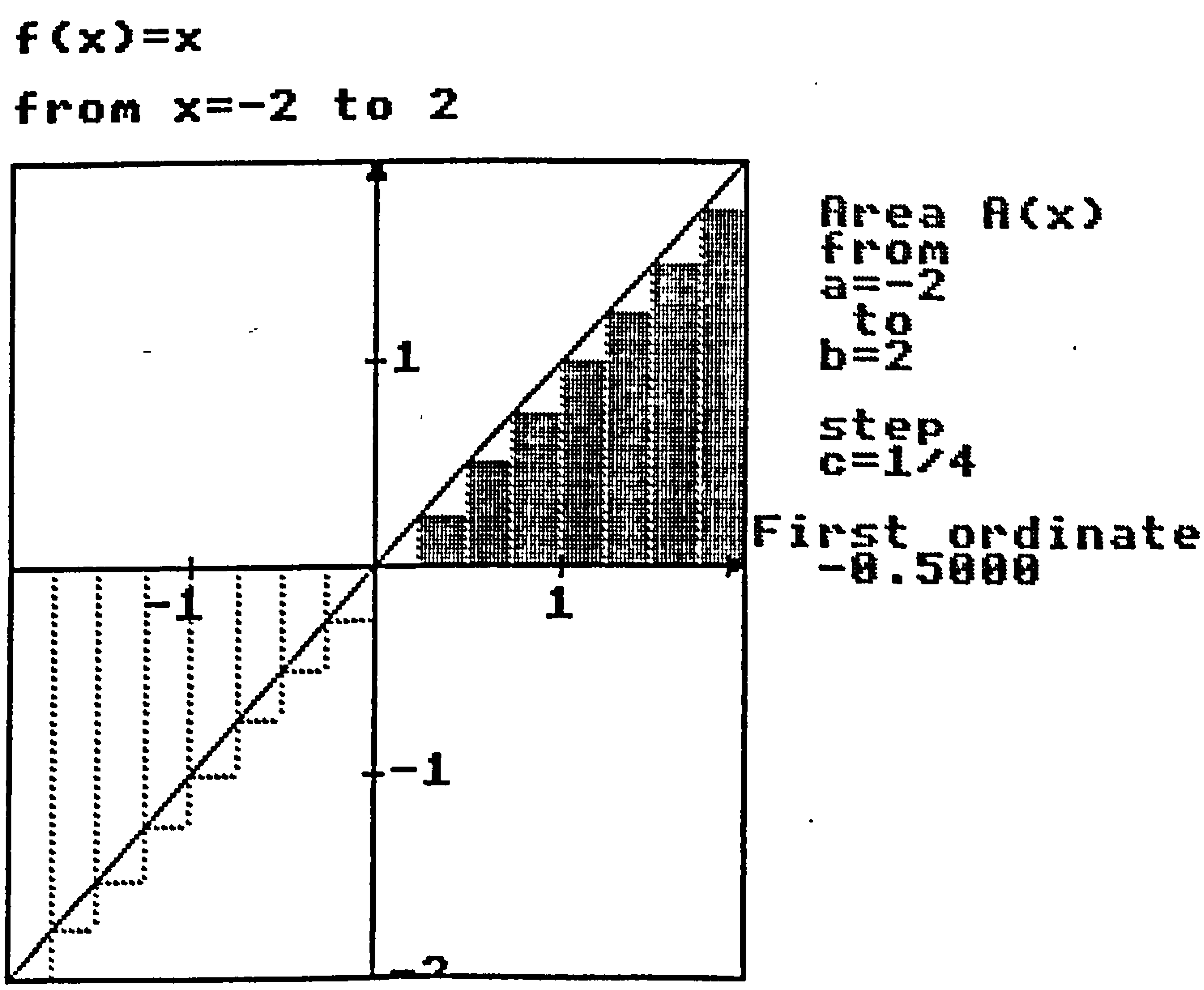



Figure 6.4

A second routine draws the *cumulative* area function. The area function A(x) under the graph of $f(x)=x^2$ from 0 to x is clearly a cubic shape (figure 6.5). By trial and error one may compare this with various scalar multiples of $x^3$ to conjecture the true value $x^3/3$.

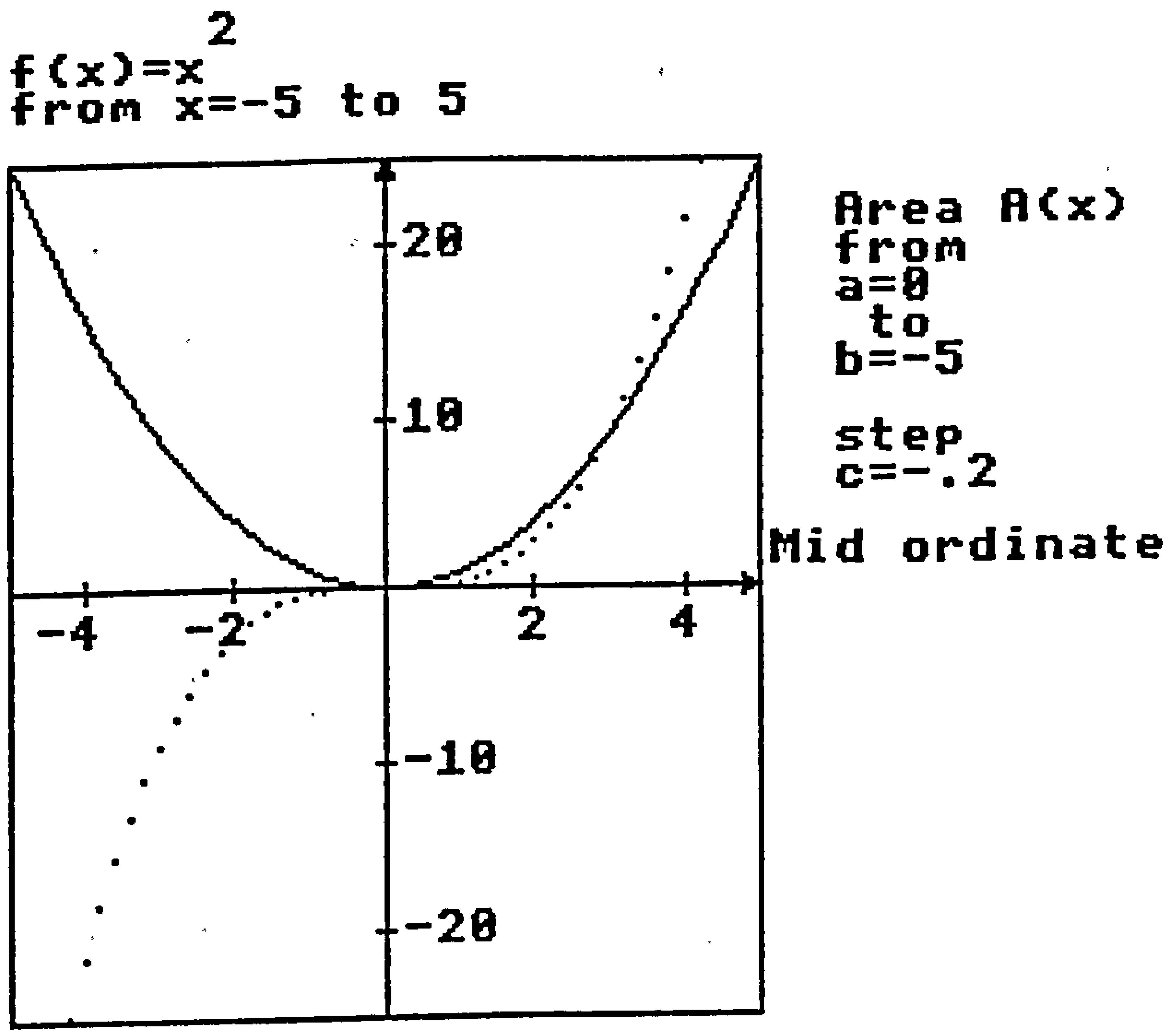



Figure 6.5

The fundamental theorem, that the area function is an antiderivative of the original function, can be demonstrated graphically in a neat way. If A(x) is the area under the graph y=f(x) from a fixed point c to the variable point x, the area from x to x+h is approximately A(x+h)-A(x). The fundamental theorem depends on the fact that

$$A(x+h)-A(x)\approx f(x)h \quad \text{for small } h,$$

with the approximation getting better as h tends to zero. Graphically this may be represented by stretching the x-range and leaving the y-range at a normal scale. If f(x) is continuous, a small horizontal stretch makes the curve less steep. The greater the stretch, the flatter the graph becomes. By applying a large stretch to a small interval this pulls out a small part of the graph approximately flat, giving a rectangle of approximate height f(x) and width h.

This is the natural place for questions of continuity to arise. Continuity is largely irrelevant in differentiation (where it is an automatic property of a differentiable function), but it arises as a separate consideration in integration. The continuity of f(x) is essential for the area function A(x) to be differentiable and satisfy A'(x)=f(x). But certain discontinuous functions have continuous area functions which are not differentiable at the points where the original function is discontinuous. The formal theory is fairly subtle but a pictorial representation of a particular example gives a striking insight. Figure 6.6 draws the cumulative area function for f(x)=x-INTx as a sequence of dots. The area function is visibly continuous, but is not differentiable at the integer points. Here the area graph has "corners" and does not magnify to look straight.

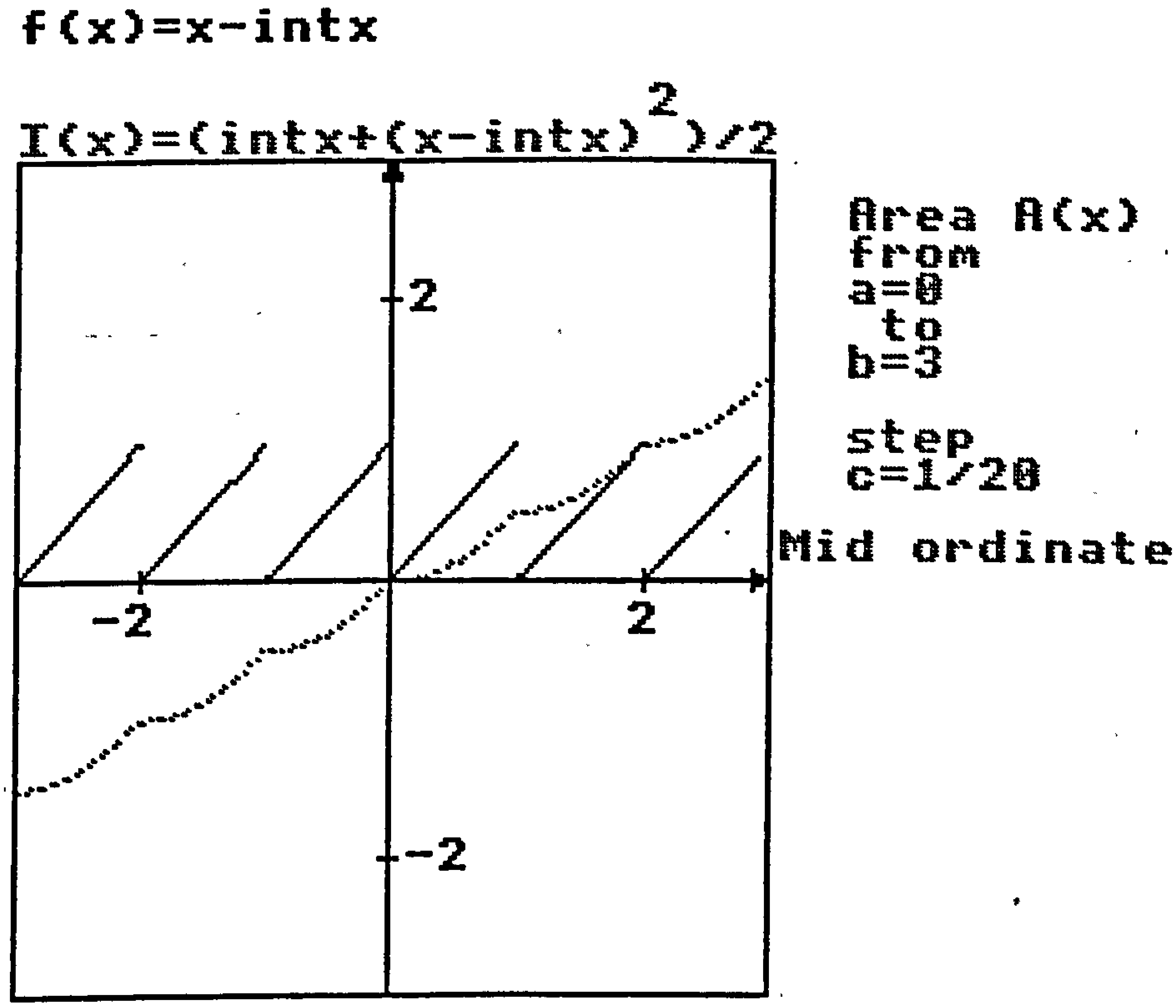



Figure 6.6

## First Order Differential Equations

In most preliminary courses the study of differential equations is no more than a rag-bag of isolated techniques for solving specific equations which happen to be amenable to a particular approach: separable, exact, homogeneous, linear with constant coefficients, and so on.

A computer-drawing approach offers a much more comprehensive view of the process of solution. A linear first order differential equation:

dy/dx=f(x,y)

is simply an extension of the antidifferentiation program mentioned earlier. At each point (x,y) in the plane we know the gradient of the required solution curve, namely f(x,y). The problem is to draw a curve which everywhere has this gradient. The naive solution is to draw a direction field with short line segments having the direction f(x,y) calculated at the mid-point (x,y). The solution is to trace a curve following through the direction field (figure 6.7).

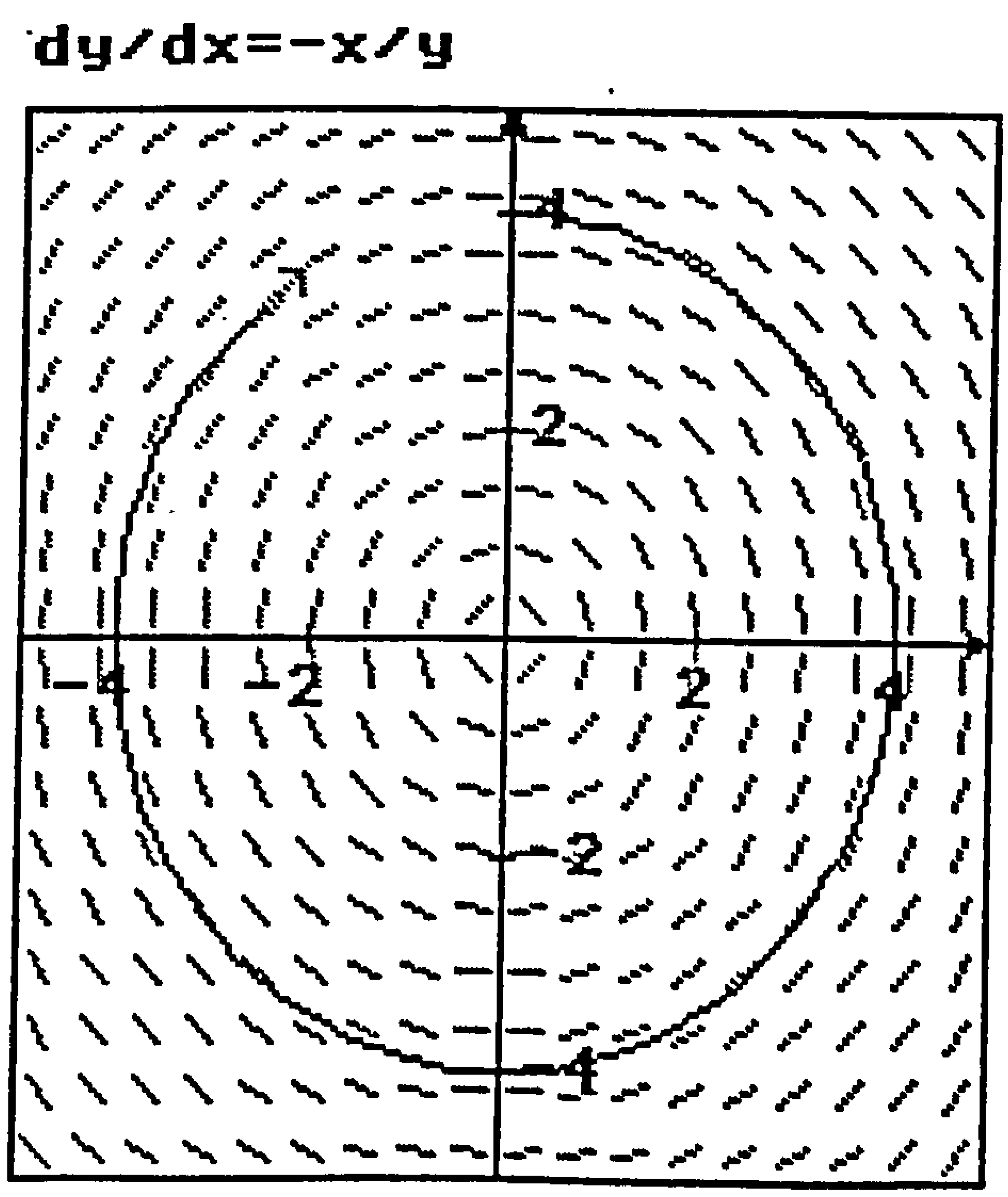



Figure 6.7

This is done numerically and may be investigated parallel to a consideration of the numerical methods involved. The picture itself powerfully suggests ideas about the nature of the solution. For instance the differential equation

$$y\frac{dy}{dx} = -x$$

does not have a global solution as a function $y=f(x)$, it has *implicit* solutions

$$x^2+y^2=\text{constant}$$

which are circles centre the origin. At points where the circles meet the x-axis the tangents are vertical. Thus the normal interpretation of $dy/dx$ as a derivative function is inappropriate, but the interpretation as a vector direction $(dx,dy)$ allows $dx=0$ with $dy$ non-zero. With the graphical interpretation it is much easier to see a first order differential equation as one giving information about the direction of the tangent $(dx,dy)$.

It transpires that several of the statements made about differential equations in elementary text-books are erroneous. It may happen that following the direction field (as in the case $dy/dx=1/x$ mentioned above) stays in a certain region of the plane. Thus the solution, and the perennial "arbitrary constant" is only relevant in this region. A global solution (for $x\neq0$)

could be $\log|x|+c$ for $x<0$ and $\log|x|+k$ for $x>0$ where k and c are different. The oversimplified statement that an "nth order differential equation has n arbitrary constants" may be seen in a more appropriate light.

To trace out a unique solution, numerically or theoretically, requires that the differential equation specifies a value of dy/dx at every point along the solution curve. The equation

$$x\frac{dy}{dx} = 3y$$

may be solved by separation of the variables as:

$$y=kx^3.$$

Every solution curve passes through the origin where the direction dy/dx is not specified. A perfectly legal solution is to have a different value of k on either side of the origin.

A combination of graphical and numerical solutions of first order differential equations gives powerful insights into the theory, complementing the isolated analytical approaches and exposing the weaknesses in the mathematics in the current curriculum.

Higher Order Differential Equations

It might be thought that the direction field in first order equations is a special case. For a second order differential equation the theory seems different. Through every point in the plane there are an infinite number of solutions. Figure 6.8 draws some of the solutions of the equation

$$d^2y/dx^2 = -x$$

through the origin.

For each starting direction from a given point there is a unique solution.

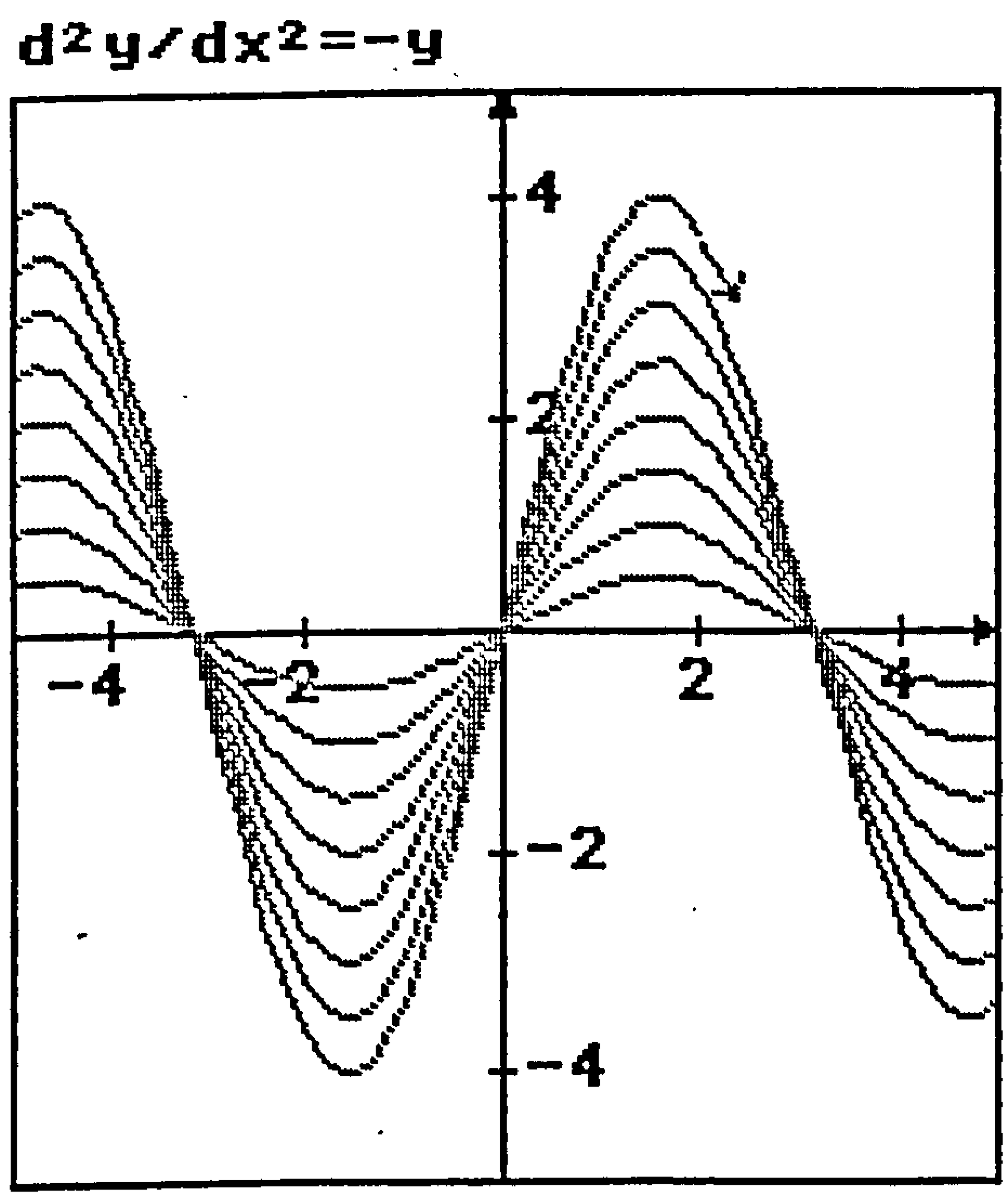



Figure 6.8

Solutions of such equations are often attacked by introducing a new variable,

$$v=dy/dx$$

giving two linear equations:

$$dy/dx=v$$
$$dv/dx=-x.$$

Thinking of this system as having one independent variable x and two dependent variables y,v then in (x,y,v) space the two linear equations again give a tangent (dx,dy,dv) in the direction (1,v,-x). Thus there is a direction field, but it is in three dimensions not two. It is possible to draw a representation of the three-dimensional solution and update appropriate coordinate planes at the same time (figure 6.9) Alternatively the three-dimensional picture may be replaced by the third coordinate plane (the "phase plane").

In this way the theory of ordinary differential equations may be given a unified meaning that enriches and complements the collection of isolated analytic techniques.

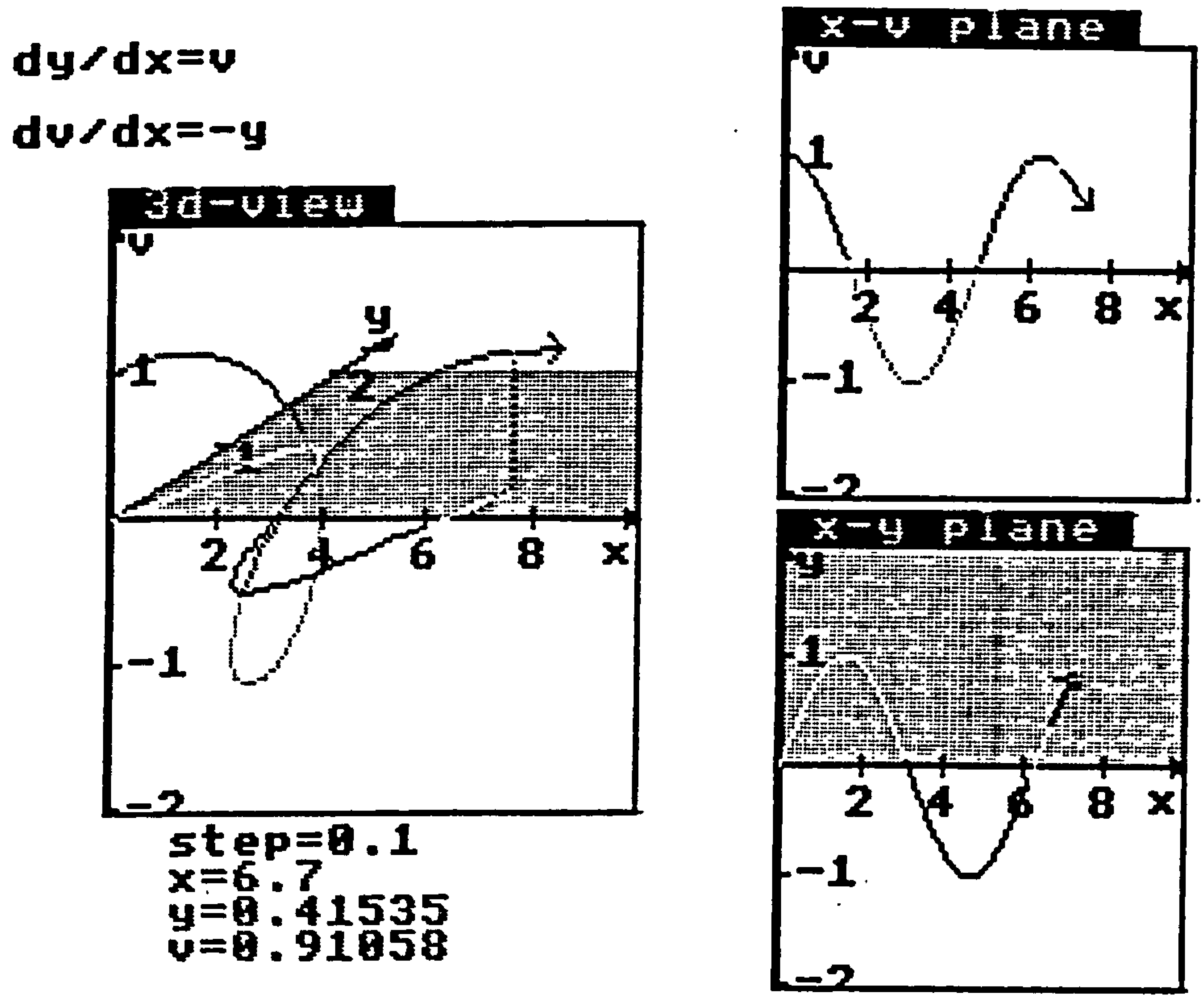



Figure 6.9

## Changes in learning style and modes of operation

The programs described in this chapter are powerful general-purpose utilities rather than self-contained programmed learning. They are structured for a wide range of uses, from teacher demonstration (with facilities to slow down or stop the action where necessary) to student investigation. Enlightened teacher demonstration can easily involve dialogue with the students rather than a straight lecture-presentation. For example, when the derivative of $x^n$ is introduced, a program may be used to draw the gradient function in various special cases. Students can *investigate* the pattern for n=1,2,3, *conjecture* the general formula for $x^n$, then *test* the formula for various values of n, such as n=4,5 or n=-1,-2, 1/2, $\pi$ and so on, before going on to *prove* the formula algebraically for certain values of n.

Often the calculus is introduced to students at a stage when the proof of the general formula is beyond them, but this does not limit their imagination in suggesting values for testing which would be far beyond their ability for algebraic manipulation, such as n=33.5 or -7/2. In drawing the graphs in such cases they may begin to appreciate the range of values for which the formulae are valid, a factor often sadly lacking in blind algebraic manipulation.

One may conjecture that students are quite capable of producing valuable results in mathematical investigations embedded in a structured curriculum. The intended legacy of teacher demonstration, discussion, and investigations is the enriched intuition mentioned in chapter 3.

# III

# Testing

## 7. Methodology and Research Design

The doctoral theses on concepts in the calculus and mathematical analysis discussed in chapter 2 (Orton [1980a], Robert [1982], Cornu [1983]) are all cross-sectional designs. The testing of the use of generic organisers in a cognitive development must be longitudinal. As the work relies on the willing cooperation of teachers, it was decided to restrict the investigation to the work done in the calculus during a single term, concentrating on the introduction of the notion of differentiation using MAGNIFY and GRADIENT.

### Ideas for research investigation

The ideas selected for investigation were developed over a period of time and have passed through a number of earlier phases. During their initial development, the computer programs were tried out in schools and certain regularities began to emerge. For example, the use of early versions of the programs for the 380Z computer showed positive student attitudes and indicated the development of their ability to visualize the gradient as a global concept (Sheath and Tall [1983]).

### Preliminary investigations

In Autumn 1983, two mixed ability calculus classes were selected at Kenilworth School out of four classes taking calculus that year. One class followed the standard calculus course without using the computer, the other followed the standard course and used the computer where possible to illustrate the concepts by computer graphics and to test out the programs under development. The experimental class were taught by the normal class teacher Mr Blackett, with the researcher (myself) present to illustrate concepts as they arose in the natural development of the given curriculum. The only new concept demonstrated was that many curves under high magnification approximated to straight lines, and this property was used as the basis of the notion of a differentiable function. Simple examples were shown of functions with different left and right derivatives at certain points (magnifying to give two line segments at an angle) and a brief look at the blancmange function to demonstrate a function which was so wrinkled, it never looked straight under a microscope. Apart from this the course followed the lines laid down by the normal class teacher, with frequent demonstrations using the gradient program and the opportunity for the students to use the computer for themselves. With only one computer available, students were divided into pairs and took it in turn to use the program to illustrate a calculation they had carried out. For example, if they were algorithmically differentiating, say, $f(x)=x^3+3x$, then they could draw this graph, use the program to draw a numerical approximation to the gradient and then compare this with the graph of the derivative they had found.

Pilot Tests: Preliminary analysis

A pre-test and post-test were given to the two groups. Results indicated that there were many similarities between the groups in the learning of calculus and certain noticeable differences. For example, those using the computer were able to visualize the derivative of a graph as a global function in a much more successful manner. Only the most able student in the group without the computer developed this facility at this stage and he was an exceptional student who subsequently distinguished himself by getting an open scholarship to Oxford.

The ability to visualize the gradient of a graph seemed to occur through familiarity with the visual process in the program, without it being explicitly taught. In virtually all other areas the indications were that to obtain clear improvement of concept imagery would require explicit assistance in formation of the concepts. Questions on the visualization of the gradient were modified for use in the pre-test and post-test in the main study. These will be discussed later in the chapter.

## Planning the main study

The following year it was intended to mount a more controlled study. This time Kenilworth School had only two groups studying A-level mathematics. It was clearly necessary to obtain further classes to participate from other schools and it was considered helpful to obtain data from students reading mathematics at university to provide a further comparison. In seeking other schools to cooperate in the research, certain obstacles became apparent.

## The constraints of the British educational system

The learning of calculus in England occurs mainly during a fairly intensive two year course for 16 to 18 year-olds in preparation for the externally assessed Advanced Level General Certificate of Education (A-level). (A number of more able students may have met the techniques of differentiation in an "additional course" in the previous year.) There are a number of independent external examination boards administering the certificate and each school may select a board, or combination of boards. Although the examination curricula are subject to modification, these changes take several years to take effect. Major experiments in curriculum change occur, such as the Schools Mathematics Project, or Mathematics in Education and Industry, but individual initiatives, such as the one encompassed in this thesis, must be framed within the system, examined by the current syllabus.

There is often insufficient time for teachers to cover the given syllabus, so that time for innovative research is bought dearly and may only be regarded of value if it has a direct pay-off in terms of improved examination performance. In particular the examinations are concerned with algorithmic manipulation of the formulae of the calculus rather than the pictorial imagery given by the computer programs. Although the latter may give the context for a fuller relational understanding, this may not be the prime objective of teachers and pupils concentrating on the mechanics of examination performance.

Any research must therefore be integrated into the system and not severely hamper the students' progress. If it takes valuable time out of preparation for the examinations it may need to be modified, or even abandoned.

Other research into new approaches to the calculus such as [Cummins 1960], or [Sullivan 1976] have shown improved understanding in certain senses without necessarily having a significant effect on performance in standard examinations. This may well be true of the current proposals.

The pragmatic approach therefore, is to take a current method of teaching and compare this with a similar approach using generic organisers on the computer. Here again there are practical difficulties. In many schools there are very small groups taking A-level mathematics, which may not provide enough individuals to use statistical methods.

As an indication of these difficulties, in 1985 there were 203 centres submitting candidates for the Oxford Delegacy's most popular A-level mathematics paper 9850. Of these centres, 70% had 20 or fewer candidates and more than half of these (42% of the total) had less than 10 candidates. There were 20% with between 21 and 40 candidates who would be likely to divide them into at least 2 teaching groups and only 10% of the centres with more than 40 candidates would be liable to have several groups. Even here the groups are often chosen on logistic grounds, dependent on choices of other subject, to fit in with a complex system of options on the time-table.

In 1985 Kenilworth School was in the middle bracket, with two mixed ability groups of students selected by their choice of mechanics or statistics. The mechanics group (16 students) became the experimental group and the statistics group (9 students) were available for comparison. However, these numbers were too small to select matched pairs for statistical comparison.

By good fortune I was able to find a larger school, Barton Peverill Sixth Form College, over 150 miles away, which offered three experimental and five control groups where the teachers were willing to keep a diary of their activities and work according to a pre-arranged plan. Regrettably, one of the experimental groups consisting of lower ability students fell behind their work schedule and the teacher withdrew them from the experiment. A corresponding low ability control group also withdrew. However, there remained more control students than experimental students covering a similar ability range and the experiment continued. As we shall see in chapter 9, there were still sufficient control students from which to select matched pairs for comparison with the experimental group, even to the extent of splitting them into subsets of those with and without previous calculus experience.

A third school that doubled as a sixth-form college and a further education college also offered to participate, however, this school only completed part of the schedule.

Two other schools offered facilities but were unable to carry the work through. At this stage one acknowledges with gratitude the willingness and dedication of those teachers who agreed to help in the research and carried it through.

## Background and Abilities of Students

Kenilworth School is in a small town within commuting distance of large conurbations in the West Midlands and has a predominantly middle-class population. There is one comprehensive school, divided into two halls for the age range 12-16 and a sixth form centre for the age range 16-18. It is not the policy of the school to teach calculus before the sixth form, although a small number of students transferring from other schools may have studied calculus before.

The two classes involved in the calculus experiment included all the students taking mathematics at A-level in their current year. The head of mathematics, Norman Blackett, was of the opinion that the two groups were fairly well-matched for ability. The classes were predominantly male, with one girl and 13 boys who eventually completed both pre-test and post-test in the experimental group, 2 girls and 7 boys in the control group.

Barton Peverill School is a sixth-form college which serves a wider area, taking students from a number of schools which cater for pupils up to sixteen. In this area it is more traditional for mathematics students to take additional mathematics including calculus for the examinations at 16. Virtually all the students at Barton Peverill had previously done a preliminary course on differentiation and integration of polynomials in the "additional mathematics" course for the London Examination board. In the first experimental group, which will be denoted by BE1, there were 5 girls, 7 boys and the second experimental group BE2 had 6 girls and 10 boys. In each the four control groups BC1, BC2, BC3, BC4 the number of girls and boys was, respectively, 9:6, 4:14, 4:10, 2:9. Overall the number of girls to boys was 12:30 in the experimental groups (29% girls) and 21:46 in the control groups (31% girls).

The third school, Cricklade College, was originally a college of further education that catered for students of a wide range of ability taking technical courses. It had recently been reorganised and now also functioned as a sixth-form college. It therefore contained students with lower ability, together with a standard sixth-form entry comparable with those at Barton Peverill and Kenilworth. Of the 51 students responding to both pre-test and post-test, 16 were girls and 35 boys (31% girls).

Its teaching methods were modelled on the higher education pattern of lectures to large audiences followed by tutorials in smaller groups. This school provided another scenario for the use of the programs, but the full plan of action was not carried out, giving markedly different results that were not used in the main comparisons of students with and without the the computer.

In addition to these sixth form classes, it was possible to give the post-test and the gradient and tangent investigations to a large class of first year university students studying mathematics at Warwick University. These were students who needed at least one grade A in A-level mathematics (achieved by between 10% and 15% of the students taking the exam). Half the class were given the post-test, the other half were given the gradient investigation followed by the tangent investigation. Although precise details of gender were not collected, the full class was approximately 30% female, comparing well with the earlier figures.

## Use of text-books

The experimental and control students in each location were studying for the same A-level examination (Kenilworth preparing for the Joint Matriculation Board, Barton Peverill and Cricklade for the London Board). In the A-level system, the syllabus, as a list of topics to be examined, is specified, the text book to be used is not. The computer programs in Graphic Calculus are therefore designed for use with a variety of different texts; they are published with a number of suggestions for use rather than as a complete study course in themselves. To simplify the design therefore, the control groups were to follow their normal practices whilst the experimental groups would use the same systems, supplemented by the generic organisers on the computer.

Work at Kenilworth

The main plan was to follow the outlines of a "teaching experiment" modifying the approach of Cobb & Steffe [1983] to meet the constraints of the British system. The main characteristics of the work with the experimental class at Kenilworth were:

(1) Twice weekly teaching (2 x 75 minutes) of a group of 16 students by the experimenter and the classroom teacher,

(2) Observation of the students' mathematical activities,

(3) Prolonged involvement with the students for a period of about 8 weeks,

(4) Clinical interviews with selected students when appropriate,

(5) A diary of classroom activities,

(6) Test papers and questionnaires.

With some regret, I ceased clinical interviews after the first sessions of interviewing, as I considered that they could give the students additional individual learning experiences that might not be typical of what can be achieved in a regular sixth-form class. The use of discussion techniques at this level is worthy of separate study.

## Work in other schools

At Barton Peverill Sixth-form College, all the teachers were willing to use a single text-book [Bostock & Chandler 1978] and instructions were prepared to indicate how the computer might be used with this text [Appendix 1]. Neither of the teachers in the experimental groups had much experience of teaching mathematics using a computer but they were enthusiastic to try. In each case they would normally have one computer for each class but a second might occasionally be available.

Cricklade College were only able to offer a limited experimental facility without any control groups, teaching all the students in a single lecture without a computer, allowing the use of computers in smaller exercise classes. The head of the mathematics department was sceptical of the use of computers at the outset and said so, although they had far better computer facilities than either of the other two schools (including a full laboratory of stand-alone BBC computers all with disc-drives).

These two schools were over 150 miles away from Warwick and, apart from a one day meeting with each set of staff to set up the plan and a further visit for a progress report, the teachers were left very much to their own devices.

## The Tests

The pre and post tests were designed to obtain information on the development of the students' concept images of certain specific mathematical concepts. These included

(1) The ability of students to handle numerical calculations of the gradient of a graph with positive and negative increments in x,y.

(2) The ability to explain the notion of calculating the gradient of a tangent, having first been asked to calculate the gradient of a chord.

(3) The use of language associated with limits and tangents. Does it coincide with that of mature mathematicians, or are there conflicting factors whose identification would assist in the teaching and learning of the subject?

(4) The calculation of derivatives of powers of x using algebraic formulae. [This question also gave an indication of those students on the pre-test who may have already done some calculus, a factor confirmed by discussion with the teachers concerned.]

In addition the post-test would look at certain aspects of the calculus which might be effected by the use of the computer and specific efforts to teach new concepts with the aid of the machine.

(5) The ability to draw the derivative of a function given graphically.

(6) The recognition of the graph of a function if its derivative is drawn.

(7) The ability to specify a function not differentiable at a given point.

(8) The abilitiy to write down an explanation of the notion of gradient of a curved graph.

(9) The ability to explain what is meant by a tangent.

(10) The ability to explain what is meant by a derivative.

(11) One question would study the meaning of the Leibniz notation as seen by the student.

All these questions, except question (3). had featured in the pilot tests, and only question (5) was modified, in this case to give a slightly easier task than the pilot question. It was a salutory experience to find that question (3) proved the most difficult to interpret, (see chapter 11), although the open-ended questions also required a great deal of effort to categorize.

During the course there would be two further investigations (some using the computer and some not) to investigate the development of the concepts of gradient, derivative and tangent. It is not a standard practice to given these concepts general definitions in a first calculus course. Rather the student comes to be aware of their meaning through usage. This leads to limited concepts and ad hoc explanations. For example, a student may consider that a tangent is "a line that touches a curve but does not cut it". To such a student a graph with a "corner" might have two gradients (different ones in each direction) and an infinite number of tangents, another student might see that it has "no gradient" and "two tangents", and so on. Thus it is likely that the students will not develop a coherent concept image of gradient, derivative and tangent which will stand the test in awkward situations. Within the constraints of the A-level syllabus their imagery is mainly concerned with gradients, derivatives and tangents of *smooth* curves so, even if they are given an explanation of the general nature of the concept, this trace in their memory may wane in the context of their experience with the standard A-level notions. In such circumstances the acquired concept image will be different from the given concept definition. The secondary investigation into gradients, derivatives and tangents is intended to give information on the variety of concept images and the feasability of giving a fuller understanding within the confines of the A-level system.

The gradient and tangent investigations were based on ideas that arose from the pilot tests, but had not been used in this precise form before.

It was planned to give the gradient and tangent investigations during the course at agreed times about a week apart (see directions in Appendix I). In the case of the university students they were given at one and the same time.

Identification of classes and students

In the following chapters the groups will be given code letters: KE, KC (Kenilworth experimental and control), BE1, BE2 (Barton Peverill experimental), BC1 to BC4 (Barton Peverill control), CE (Cricklade experimental), U and U2 for the two halves of the university class. Students will be given additional numbers, so that BC103 is the third student in BC1, and those with calculus experience before starting the experiment will be marked with an asterisk, for example KE07*.

## 8. Experiences in the classroom

The main teaching experiment consisted in introducing the generic organisers for the derivative into the existing mathematics classes. At Kenilworth Castle School, Mr Blackett agreed to teach his class using his standard techniques but, whenever the use of the organisers was appropriate, he sat at the side of the class and allowed me to introduce them. Meanwhile, a second class taken by Mr Derek Morris followed the course only using the textbook. Material was selected by the two teachers from Dakin & Porter [1980] and SMP Book 1.

In Barton Peverill Sixth Form College, five mathematics classes were involved, basing their work on Bostock and Chandler [1978]. All the teachers worked through the text (chapter 5) with the two experimental classes having the additional facilities of a single computer available for demonstration and a second sometimes available for student experimentation. The teachers followed instructions prepared to indicate the use of the computer (Appendix 1) and kept a diary of their activities. The instructions were to follow the normal teaching pattern, but those using the computer were to broaden the notion of gradient of a graph by looking at examples and non-examples of the concept, and to attack the notion of tangent using the computer. The teachers of experimental groups were asked to include the following specific teaching aims, in additional to their normal pattern:

1. The computer should be available at all lessons and used by the teacher for demonstration and

2. by the students for exploration whenever possible. (Suggestions were given as to how this may be done: see appendix 1.)

3. In addition to the work in the text, the notion of GRADIENT should be emphasised as the gradient of the *graph* itself, by demonstrating that a differentiable graph highly magnified looks straight.

4. Examples of *non*-differentiable functions should be given to set the concept in context. These do not magnify to look straight (such as absx at the origin, abs($x^2$−1) at x=−1 or +1, or $2^{absx}$ at x=0).

5. The notion of tangent should be approached through using the computer to draw a line through two nearby points (using SUPERZOOM).

6. Examples of functions which do not have tangents should be given to set the concept in context (as in 3).

7. The link should be made that if a function is differentiable then it has a tangent, and vice versa.

Notice that no specific instruction was given to show the students how to sketch the gradient of a graph; it was hoped that this ability would be a consequence of using the generic organiser.

The same practices were followed in outline at Kenilworth, but here the texts available were slightly less suitable for use with the GRADIENT program, so a two page description of the notions of derivative and tangent were given to accompany the programs in the initial stages.

At Cricklade College the teaching method is by large lectures followed by smaller tutorial groups. This will be the subject of separate comment later on.

Much as one would wish for more time for exploration, the scheduled time for lessons in Kenilworth and Barton Peverill very much followed the SIMs report [McLean et al. 1984] in Ontario with the majority of the time used by the class working as a group. When the computer was absent, the standard lesson in Kenilworth started by tidying up any detail left over from the previous lesson (including reviewing any homework), teacher instruction with questions directed at the class, followed by the class doing examples with the teacher walking round checking work and answering individual queries. With only one computer available the hope was to introduce more two-way discussion using the computer as a focus and to initiate its use by students wherever possible.

### The Kenilworth diary

<u>Lesson 1</u> (9.10.84, 75 minutes)

The class were given the pre-test (see Appendix 2) for about half an hour. The program MAGNIFY was made available for students to experiment with as they finished the test. Alan (KE07*) finished first and was told to use the program to draw some graphs of his choice, then magnify parts of the graphs to see what happens. He drew $y=1+x^2$. His comment was that, on magnification the graph looked "less curved". By this time, several others joined him and they took it in turns typing in graphs. Initially, they seemed awed by the task and found it quite difficult to think of functions to draw. At my suggestion they opened a mathematics book and selected formulae at random. They tried $y=x+1/(x-1)$ first, then moved on to others to see if the idea of "less curved" always worked under magnification.

By this time the whole class of sixteen were free with about forty minutes to go. We proceeded with a class discussion under my leadership, first looking at more functions under magnification, taking suggestions from the students themselves. Each time the student making the suggestion was invited to type in the function and suggest suitable ranges for drawing. In this way they were familiarizing themselves with the use of the program and the method of input. There were one or two more enterprising suggestions, such as logx and $2^x$.

There seemed a general belief that "magnified graphs" would look straight, though Adam (KE14) suggested (very perceptively) that if the magnified picture was as large as a football pitch, the *whole* picture wouldn't look straight, it would be just the original curved picture on a larger scale. We agreed that we actually meant that a *suitably small part* of the graph highly magnified might look almost straight.

This is an example of what constructivists call *"negotiation of meaning"*; using the computer to explore the meaning in this way proved a very valuable exercise, allowing students to express their doubts and take part in their own formulation of the concept under discussion.

The students were asked if *all* functions would look straight if a suitably small part were magnified. Ian Wells (KE16*) was sure they would, but when asked to give evidence in support of his assertion, he was less sure. None of the class could suggest a graph that did not magnify in this way, though one student said he could imagine that a graph might not look so smooth but he couldn't think of a formula for it.

At this stage I suggested the graph y=abs($x^2$-1), drew it, and showed how the absolute value (or modulus) was responsible for taking the graph of $y=x^2-1$ and "flipping it over" where this graph was negative, always to give a positive (or zero) value. I asked what happened at x=-1 or x=+1. The class were silent. They seemed not to know what was required of them. A "volunteer" was chosen to magnify the graph near x=-1. The magnified picture revealed two half-lines meeting at an angle (figure 8.1).

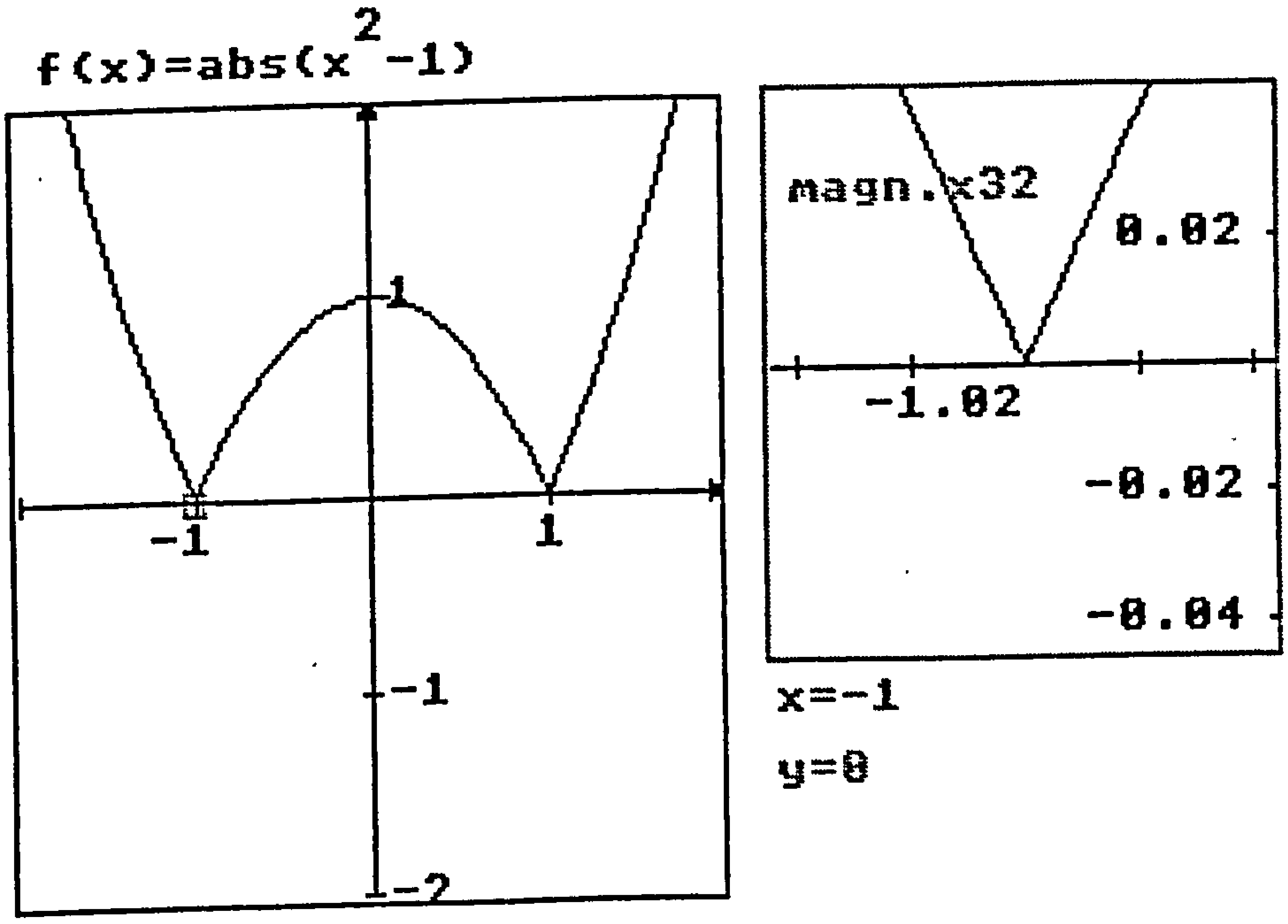



Figure 8.1

In the few minutes before the end of the lesson there was time to look briefly at the blancmange function: how it was built up graphically and what parts of it looked like under magnification. Some graphs are so bad that they nowhere magnify to look straight!

<u>Lesson 2</u> (11.10.81, 75 minutes)

This lesson was intended as an introduction to the notion of the gradient of a graph, building on the magnification experiences of the previous lesson. The textbooks available were SMP [1967] and Dakin and Porter [1980]. Mr Blackett considered the explanation using "scale factor" in SMP "too wordy" so, following his advice, I used three examples from the SMP chapter, then switched to Dakin and Porter's more traditional approach, supported by the computer.

SMP p.88 Q.2 gives a distance-time table at 5 second intervals; the exercise is to estimate the average speed over various periods and to give an estimate of the speed at a particular time. This was done as a class exercise without the computer, then the students were set two similar questions (5,6 page 89), cooperating amongst themselves if desired.

The idea of "rate of change" was then explored on the computer. I drew the graph of $y=x^2$ (ranges x=-2 to 2, y=-2 to 2) and discussed the rate of change as the "y-change" over the "x-change", which was a concept with which they were familiar for straight lines. As our experience in the previous lesson showed that the graph was approximately straight over extremely short sections, perhaps we could get the gradient near a point by working out the "y-change" over the "x-change" for nearby points. I suggested the students concentrated at the point x=1/2, and imagined what the gradient would be there. The consensus from several members of the class was 1, or thereabouts. We went through the routine, selecting a=1/2, b=1.5 to draw the extended chord through $(a,a^2)$, $(b,b^2)$ and getting the computer to calculate the gradient. Then the routine was used to let b move in steps to a, displaying the chord-gradient at each step (figure 8.2). The chord-gradient got closer to 1 until, for a time, it was displayed constantly as 1.0000 (to four decimal places), so that the "nearly straight" curve had a gradient approximately 1. I asked if it was *exactly* one, and the response was "not quite, but the error was less than four decimal places".

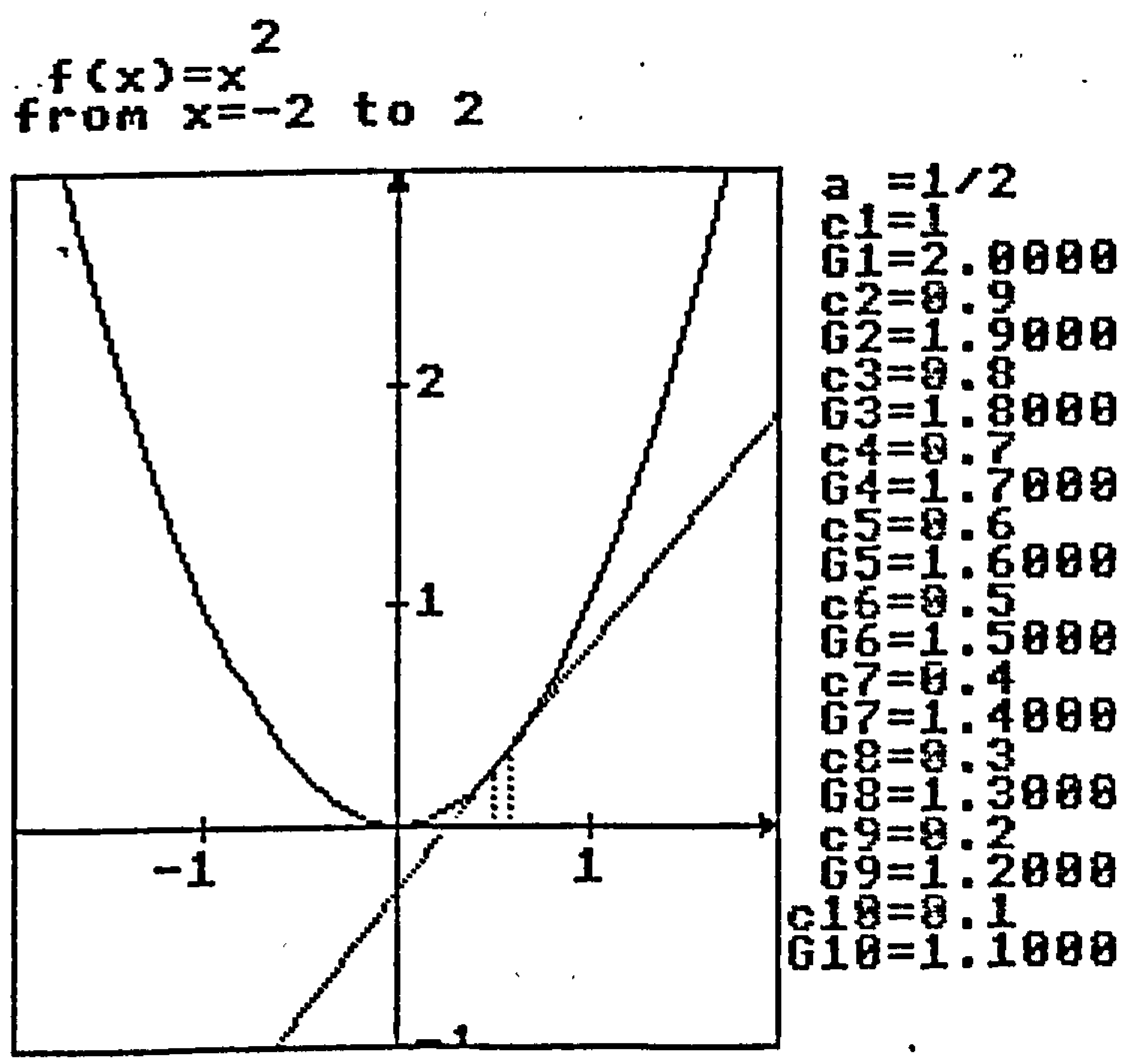



Figure 8.2

We then considered what happened at $x=2$, and at this stage I asked the students to open Dakin and Porter [1980] at page 28, where the calculation was carried out numerically in the case $x=2$ to get a chord-gradient approaching 4. We let the computer demonstrate this limiting process in its own way.

The students were set the problem of calculating the chord-gradient from $x$ to $x+h$, starting from the formula

$$\frac{(x+h)^2-x^2}{h}$$

The end of the lesson was taken up using GRADIENT to illustrate the global chord function. First $c=1$ was used to plot chords of gradient $(f(x+c)-f(x))/c$, slowing down and pausing in the process to explain what was going on.

The gradient graph was a (almost) a straight line, give or take a pixel. The chord gradient function for $c=1/100$ was then drawn. As it clicked along the process was again paused to discuss what was happening. Adam (KE14) remarked that the chord looked like a tangent.

When invited to suggest its formula, Ian Pringle (KE11) suggested $2x$.The gradient function now looked almost like the line $2x$. A volunteer typed this in for comparison and the suggestion was confirmed. The fact that the "straight line" on the computer screen was not perfectly straight did not seem to be a factor in the discussion.

Lesson 3 (16.10.84, 75 minutes)

The lesson began with the students attempting Dakin and Porter exercise 3 question 1, to perform numerical calculation of limits of chord-gradients for $y=2x^2$ from $x=1$ to 1.1, 1 to 1.01, ..., 1 to 1.00001 from first principles. This proved quite time-consuming, both understanding the task and filling out a table of values using a calculator.

The students seemed pleased to see the same idea carried out easily by the computer, simultaneously displaying the calculations and drawing the extended chord.

At this stage I handed out a two page summary explaining the notion of derivative and the linked idea of tangent.

This reworks the algebra for calculating the derivative of $f(x)=x^2$ and gives a "dynamic limit" explanation of the gradient, linking it to the magnification property. A brief discussion of the tangent to a curve at a point P is given as:

1. It is a specific straight line $y=mx+c$ through $P=(x_0,y_0)$
2. If Q is the point on the graph with x-coordinate $x_0+h$, then as h tends to 0, so the gradient of the chord PQ tends to m.

The summary goes on to discuss the tangent as the "limiting position" of the extended chord, explaining that the term "limit" is used in a sense different from that in ordinary life. It explains that a graph "can only have a gradient or a derivative or a tangent at a point if it magnifies to look straight there".

This document looks very much like the formulation found in many textbooks, plus the added notion of "magnifying to look straight" and a more refined definition of the tangent. However, we had the added advantage of being able to have a class discussion using the computer to draw pictures and carry out calculations that would prove too time-consuming by hand (or by calculator).

At my suggestion a volunteer typed in the formula $y=abs(x-x^2)$, into the program GRADIENT. The graph drawn by the computer clearly had different left and right gradients at the origin. These were calculated numerically from the right and left by the program, demonstrating the limiting procedure from each side.

I asked what would happen if the computer were used to calculate the gradient of $y=x^2$ at $a=1$ from the left or right. The students correctly predicted the result would be gradient 2 from either side. The same calculation was carried out with the graph magnified over the x-interval 1-1/100 to 1+1/100 where it looked straight on the VDU. As the chord was drawn through $a=1$ and the point b moved nearer a, the chord and the original graph were drawn on top of each other.

The idea was reinforced that, although several lines might seem to "touch" a graph at a corner, the mathematical definition of a tangent only applies when a small portion of the graph magnifies to look straight. Ian Pringle (KE11) was very disturbed by this. He had come from Scotland where he was taught that a tangent just "skims" the graph, so a graph could surely have lots of tangents at a "corner".

I loaded SUPERZOOM into the computer and used it to draw the tangent first to $y=x^2$ at the point $x=1$, where it gave a satisfactory picture, then with the graph $y=abs(x-x^2)$ at the origin, where it gave the "right-tangent". I explained that it didn't draw a theoretical tangent, because the computer program only had the formula for the graph and the program was not capable of manipulating the formula to obtain the formula of the tangent. What the program did was to use the formula for the graph to calculate two points on the graph for nearby x values, x and x+.00001, then draw the line through these points. I used SUPERZOOM to draw the line through the points on the graph with x-coordinates 1, 1.00001. It gave a graph which seemed to touch the graph to the right. Similarly the graph with x-coordinates 1, 1-0.00001 seemed to touch the graph to the left.

Adam (KE14) suggested these weren't fair: we should try the graph through 1+.00001 and 1-.00001 and we got yet another line, this time almost horizontal.

I reasoned that the only way that the graph could be said to have a specific gradient at a point was when it magnified to "look straight", and then it had a single tangent. If it had different left and right gradients, or worse, if it were like the blancmange function, all wrinkled, then we wouldn't get a unique tangent. The lesson finished with the link between gradient, derivative and tangent being reaffirmed.

The next lesson was the half-term test on algebra, done earlier, and was followed by a week's holiday. We returned to the calculus a fortnight later.

<u>Lesson 4</u> (30.10.84, 75 minutes)
The plan was to revise the previous ideas briefly and to introduce the $\delta x$, $\delta y$ notation with the quotient $\delta y/\delta x$ tending to $dy/dx$, as in Dakin and Porter. I also showed how dx and dy could be interpreted as x and y increments to the tangent respectively, but this proved to be brief and there was no time to follow it up until after the post-test.

Using Dakin & Porter page 33 exercise 3, I demonstrated the algebraic calculation of the chord gradient of $y=x(x-2)$ from x to x+h, considering the limiting process as h tends to zero. The students were asked to do exercise 4 to calculate the formal gradients of the curves $y=x^2+x-1$ at (1,1) and $y=1/x$ at (1,1).

Some students found difficulty here, for example Naomi (KE01) did not know how to substitute x+h for x and requested help.

After a check that all students were satisfied with their calculations, we turned to the bookwork pages 29-32 which uses the $\delta$ notation. The first part was revision of the gradient on $y=x^2$ using $\delta x$ instead of h. Next the curve $y=x^2-2x$ was sketched on the computer and the calculation carried out on the board to find $\delta y/\delta x$ to be $2x+\delta x-2$. For small values of $\delta x$ this approximated to $2x-2$.

The "chord gradient" option in GRADIENT was used, first with c=1 to see what was happening, pausing in the process to see the moving chord and its gradient simulaneously being plotted. Then the process was repeated for c=1/100 and the chord gradient compared with 2x-2. From now on, as students calculated the chord gradient from first principles using the formula, it was also plotted using the global gradient option in GRADIENT.

The students then started to do exercises, calculating simple derivatives using the $\delta$-notation then using the computer in turn to draw the graph of the chord function to see if they had the right answer. They were divided into six groups of two or three students to take it in turns using the computer. The plan had been to attack Dakin & Porter, pages 34-5 exercises 5-8, 10-16. Regrettably the end of the lesson arrived with only a couple of questions finished and only two groups of students having used the computer themselves. The rest were set as homework.

## The Open Evening

An opportunity presented itself for leisurely exploration at a parent's evening. Approximately half the class volunteered to take part in an exercise. The plan was to work through an investigation in small groups to discover the gradient functions of a number of graphs using GRADIENT.

It began asking them to use the computer to draw $f(x)=x^2$ and the chord gradient function, then to conjecture the formula for the gradient, as they had done in class. It then asked them to investigate what would happen with $f(x)=x^2-1$ and $f(x)=x^2+1/2$. Next they were asked to guess the gradient function for $f(x)=x^3$, then to conjecture the gradient function for $f(x)=x^n$ and to test their conjecture for other values of n (e.g. n=4, 1/2, -1). This was followed by exercises to guess the gradient function of sinx and cosx (angles in radians), ln(x) and ln(abs(x)). The final question was to find a value of k such that the gradient function of the graph of $k^x$ was again $k^x$.

I expected the work to keep a group of students going for half an hour or so. They were encouraged to get on with the investigations themselves and I sat nearby to answer any technical questions about the program.

They needed little encouragement. The first group of three students, lead by Ian Pringle (KE11), with Colin (KE09) and Adam (KE14) participating occasionally, sped through the sheet. They felt the first exercise, to see what happened when various constants were added to $x^2$, was rather trivial because it clearly didn't change the gradient to move the graph up by a fixed amount. By building from the case $f(x)=x^2$, which they knew, they soon guessed the gradient function for $x^3$ at the second attempt (after trying $2x^2$) and thought that the gradient of $x^4$ would be $4x^3$ from the "general pattern". The formula for the derivative of $x^n$ was suggested as "n x to the n minus 1" (though Ian didn't know what the word "conjecture" meant). They tested the formula for n=-1 and n=1/2 and found it still worked. Almost as quickly as they could type in the formulae they correctly conjectured the derivatives for sinx, cosx and suggested that the derivative of $k^x$ was again $k^x$ for k approximately equal to 2.7. The gradients of ln(x) and ln(abs(x)) were seen to be 1/x with little difficulty and they had to be given the extra challenge of tanx, which was guessed as $1+(\tan x)^2$ using the reasoning that it "looked a bit like the square of the original graph, shifted up by 1".

Appearances are deceptive here. Adam enjoyed being part of the team, even suggesting things to do, but he confessed to me after that he wasn't sure what was going on and asked for more explanation of how the computer worked out the gradient.

The second group of three, in which Simon (KE08) was prominent, worked more slowly, but almost as surely. The gradient of $x^3$ was first thought to be $x^4$ then $2x^2$ and then, finally $3x^2$, by trial and error. The formula for the derivative of tanx was guessed fairly easily. They had a little difficulty with the derivative of ln(x), thinking it ought to be "something to do with ln(x)", trying $\ln(x)^2+1$ before alighting on 1/x.

The group continued after finishing the given tasks by drawing various functions with the GRADIENT program to "guess the gradient". At the end Simon had typed in $f(x)=(1-x^2)^{1/2}$, to try to guess the gradient for a circle; he failed. (In two subsequent lessons he was to come into class and try other guesses, none of which worked, but this was to provide a valuable stimulus later when we came to the chain-rule where we were to use a substitution to produce two functions $u=1-x^2$ and $y=u^{1/2}$, both of which he could handle.)

The students were able to benefit from their knowledge of graph-sketching which they had studied in detail for polynomials earlier in the term. It transpired that they had not yet done trigonometric functions in radians in mathematics, though it had been done in physics.

<u>Lesson 5</u> (1.11.84, 75 minutes)

The lesson began with a review of the bookwork on differentiation from first principles using the Leibniz notation and the rules for sums and constant multiples of functions. The students then carried on with formal exercises from Dakin and Porter, pages 34,35. They were divided into six groups of two or three, and each group assigned two exercises. They did the all the formal exercises in order, but as each group finished their assigned ones, they were encouraged to type it into the computer to check that their formula truly gave the gradient of the graph.

<u>Lesson 6</u> (6.11.84, 75 minutes)

Before Mr Blackett arrived for the official start of the lesson, Graham (KE05), explained that he had been told in physics that "the derivative of sinx is cosx" without explanation and asked to be shown why this was. He was not present at the investigations at the Open Evening.

He was invited to the front of the class to type in sinx from $-2\pi$ to $2\pi$ and and to draw the approximate gradient function using the chord method. He immediately smiled and said "OK". I suggested someone else try to guess the derivative of cosx. The suggestion "sinx" came and the "volunteer" took his turn to try it out and change his mind to "*minus* sinx". Why? "Because the graph is upside down..."

Brian (KE15*), who had recently come from a well-drilled Public School, sat at the back of the class and expressed disgust. He couldn't see why we had to guess these things. Why couldn't the teacher *say* what the derivatives were, so that he could *learn* them.

At this stage Mr Blackett arrived and the lesson began. I took the lead once more, looking at the gradient of graphs that some already knew. The rest were invited to guess the derivative of $x^3$, typed in by a volunteer, then its gradient function drawn with c=1/1000. Brian was invited to suggest a formula. He first suggested $x^2/2$, then $2x^2$ and, finally, $3x^2$. The class *was* asked to work this out by formal algebra using the dx notation. It was carried through with a little mutual support between the students. On being asked for the derivative of $x^n$, the formula $nx^{n-1}$ was suggested in the form "you put the power in front and decrease the power by 1".

I gave an outline proof on the board, using the long division method:

$$\frac{a^n-b^n}{a-b}$$

$$=a^{n-1}+a^{n-2}b+...+b^{n-1},$$

so that, when b gets close to a, this gets close to $na^{n-1}$. I was not sure that it was fully understood.

I suggested that they might like to try another value of n in the computer program to check if it worked, expecting a reasonable generalisation, such as n=5. Graham (KE05) suggested n=33.5. The program coped.

At this stage I suggested that the earlier demonstration of the formula only worked if n was a whole number. Was it necessary to "prove it" in general? The consensus was that it wasn't necessary, one suggesting "it is obviously true..." I explained the difference between a mathematical proof, where we needed to show that the result was true in all cases without exception, and a scientific proof, where we tested the truth in specific cases. Despite this I did not feel that many of the students sensed that a proof was necessary.

We looked at n=-1 and n=-2 on the computer, recalling that $x^{-1}=1/x$ and $x^{-2}=1/x^2$ and the students were invited to do the calculation of the derivative using the $\delta$-notation.

Again the results were true, which only seemed to confirm the student's feelings.

At this stage the lesson ended and the homework was to read the bookwork starting chapter 4 and to tackle examples 4a questions 1,2 which were examples of differentiation from first principles and simple use of the rules derived in the bookwork.

<u>Lesson 7</u> (8.11.84, 75 minutes)

Mr Blackett began the next lesson, first going through any difficulties in the homework. The students had little difficulty differentiating polynomials from first principles, but 1/2x caused some problems with algebra. He went through this and went on to talk about the use of different letters in differentiation, e.g. ds/dt instead of dy/dx, ready for some of the later examples. The students were again divided into small groups and assigned specific questions they could type into the computer as they solved the problems. There was a fairly orderly use of the computer as they calculated the derivatives in the following exercises. All the groups managed one trip to the computer but not all fitted in a second.

In the last part of the lesson Mr Blackett moved on to look at a few examples of calculating tangents (Dakin and Porter page 44). This was concerned mainly with the techniques of finding the tangent and performing simple coordinate geometry.

<u>Lesson 8</u> (13.11.84 75 minutes)
This was taken by Mr Blackett, beginning with the ideas of maxima and minima. He introduced the ideas and explained that dy/dx=0 at a local maximum or minimum, sketching the case $y=x^3$ on the board to demonstrate the possibility of a local inflection. He also warned that a maximum or minimum on an interval could occur at an endpoint, saying this was a favourite "trick" question on examinations. He asked the students how they might tell if there was a maximum or minimum from a changing gradient, moving his hand through the air up and down. Andrew (KE10) suggested that

the gradient would be positive before, and negative after, a maximum, and there was general assent from the rest of the class. Mr Blackett also demonstrated negative before, and positive after, for a minimum.

I interposed, suggesting that a graph like y=abs(x) might have a minimum with a "corner" and here the graph didn't have a derivative, because it didn't magnify to look straight. Thus we must know first that the graph has a derivative at a maximum or minimum at a point to be able to assert that it was zero there. I initiated a discussion on problems of maxima and minima in linear programming, but this was rather abortive and inappropriate.

Mr Blackett ran through a couple of problems involving maxima and minima; about forty minutes were left for the students to do examples from SMP page 104, questions 5 to 8 and Dakin and Porter page 74, questions 4,5,6. As I attended to the students difficulties, a number of errors in algebra and differentiation came to light. Adam (KE14) tried to differentiate

$$\frac{(300T-2400)-100-5T}{T}$$

"top and bottom", Gareth (KE10) made the mistake of writing this as

$$(300-2400/T)-100+5T,$$

and Brian (KE15*) managed to do all the calculus and algebra in question 7 to end up with the equation

$$8x^3=729,$$

which he could not solve.

During this time the students were invited to use the computer to draw the gradient function to look at maxima and minima in any of the questions. Only two groups of two (four students in all) did so.

The students were invited to finish off the examples for homework.

<u>Lesson 9</u> (15.11.84, 75 minutes)
The students were asked to do the "gradient investigation"; they managed in about fifteen minutes. They were invited to use the computer, but none of them wished to do so.

Following this, Mr Blackett went over difficulties in the homework. There were some examples that were causing problems. Adam again was using inappropriate "rules of differentiation", this time he tried to differentiate $V=\frac{1}{3}\pi r^2(6-r)$ by differentiating each part of the product to give $dV/dx=\frac{2}{3}\pi r(-1)$. His action was a perfectly reasonable inference: the formulae worked for addition and constant multiplication, why not for multiplication of functions? After all, he had not been told that differentiation of a product in the obvious way would *not* work.

This time I went through the "product formula" with him in the $\delta$-notation. Although I was not sure he fully understood, he was relieved to see there was a reason for his error and went back to solve it by first multiplying out the bracket.

Homework was to finish off exercises on maxima and minima from Dakin & Porter page 74.

Lesson 10 (20.11.84 75 minutes)

Mr Blackett checked the homework and found David (KE04) couldn't differentiate

$$\frac{\sqrt{3}}{4}y^2+9y-\frac{3y^2}{2}$$

because "I was away when you did roots". Even when it was pointed out that $\sqrt{3}/4$ was a constant, he remained stuck. Colin (KE09) couldn't do it either until he was heavily prompted by Mr Blackett. He read $9y$ as "nine y to the nought" and couldn't differentiate it. Mr Blackett revised differentiation of expressions such as $6y^3+4y^2+3y-\sqrt{y}$ and $3y^2+2y+y+1/y+3/y^2$, but it was clear that some students were having difficulties with strange constants, e.g. $\sqrt{3}/4$ and some were finding difficulties with $x^n$ for negative and fractional $n$.

These difficulties in algebra clearly relate to the phenomena noted by Küchemann (in Hart [1981]) which are worthy of further study.

At this stage Mr Blackett moved on to trigonometric functions, looking at angles in radians, at the ratio $\theta/\sin\theta$ as $\theta \to 0$ and getting the students to draw the graphs of sine and cosine by hand (see Tall & Blackett [1986]).

<u>Lessons 11-15</u> (each 75 minutes)
These were all taken by Mr Blackett, covering ideas on trigonometric formulae, getting students to draw graphs such as sinx, cosx, 2sinx, sin2x, $\sin^2 x$ by hand, then using the computer to confirm the picture and sort out difficulties. The trigonometric formulae for cos(x+y), sin(x+y) were demonstrated by matrix multiplication. There seemed to be a note of tension, relieved only when the formulae were finally written up in a form that could be learnt. These were confirmed with SUPERGRAPH, drawing graphs such as sin(x+a) and comparing this with sin(x)cos(a)+cos(x)sin(a). There followed formal exercises solving equations involving trigonometric formulae, further standard formulae such as those for sin2x, cos2x, tan(x+y) etc., then more equations to solve, such as writing acosx+bsinx in the form $r\cos(x-\alpha)$ and solving acosx+bsinx=c. Finally equations were considered of the type $\sin^2\theta+3\sin\theta-2=0$ (quadratic in $\sin\theta$) and $5\sin^2\theta-\cos\theta+2=0$ (requiring a single modification using a formula

to give a quadratic in a trigonometric function).

The structure of all these lessons was: check homework, class teaching with directed questions to individuals, followed by class exercises and help for individuals.

Lesson 16 (11.12.84, 75 minutes)

The derivatives for sine and cosine were explained by Mr Blackett using the forms

$$\frac{\sin(x+h)-\sin(x-h)}{2h}$$

and

$$\frac{\cos(x+h)-\cos(x-h)}{2h}$$

The computer was available, but it was not used initially. The idea that lim (sin(x+h)-sin(x-h))/(2h) would give the same result as that using lim (sin(x+h)-sinx)/h was underlined by reference to earlier ideas on magnification, so that if the graph was almost straight in the small, one would get approximately the same values for the gradient between x and x+h as between x-h and x+h. For smaller values of h the approximations would get better.

The derivatives of sin(kx) and cos(kx) were treated graphically on the board and then demonstrated on the computer.

Lesson 17 (13.12.84, 75 minutes)

The school end-of-term test was administered, followed by the test on tangents. Although the computer was available for the tangent test, no student felt it necessary to use it.

Lesson 18 (18.12.84, 75 minutes)

The post-test was administered.

Comments on the Kenilworth Experience

Reviewing the records of the Kenilworth experimental class, it will be clear that, apart from the Open Evening, very little individual exploration occurred. With as many as sixteen in the class and only one computer it proved difficult to give every student adequate personal access to the computer during class time. However, the evidence here and the repeated experience in the class exploration of the gradient of $x^n$ showed that GRADIENT could clearly be used to investigate the derivative of this formula, expecting some measure of success. Other functions such as sinx, cosx, $e^x$ and ln(x) could be fruitfully investigated in the same way.

The basic scheme used was the standard format of teacher demonstration, followed by individual exercises with the teacher monitoring progress and helping students. However, it seemed that the atmosphere of the class during the use of the computer was more conducive to mutual discussion between the teacher and the class. Without the computer at all, the pattern of formal lessons was likely to predominate with both teachers concerned.

The effect of the examination system was felt strongly in the planning of the lessons. Limited time meant that hoped-for study of concepts, such as further discussion of tangents, or of the dy/dx notation, or more examples of non-differentiable functions, were squeezed out. The need to get through the required number of techniques, to encounter all variations of difficulties that might occur in formal manipulation on the examination paper, proved extremely strong.

One might conjecture that within the current system, a successful cultural element, such as the use of the GRADIENT program to investigate standard formulae, might survive, but more esoteric ideas (to the average teacher) of giving examples of non-differentiable functions at the outset, might not withstand the need to "get on with the syllabus".

Barton Peverill Experimental Group 1

The students at Barton Peverill differed from those in Kenilworth in that the majority already had experience in calculus techniques, but not in graphic visualisation of the concepts.

The researcher was not present at the lessons (except for a single visit, mentioned later). The report of the events given here is based on lesson diaries kept by the teachers, supplemented by discussions which took place afterwards.

The teachers in both experimental groups went through the suggested order given in Appendix 1. Initially there were hardware difficulties with the monitor, so a smaller monitor had to be used so that some students could not see the writing on screen. This did not prevent some use being made for class demonstration or individual use by groups of students.

In experimental group 1, the pretest was followed by looking at the graph $f(x)=x^2$ and calculating the gradient from the left and right of a=1, a=1/2, in the latter case under high magnification. In the second session the students worked in five pairs and one threesome to calculate the gradients in Bostock and Chandler, Exercise 5a. Session 3 was concerned first with the delta symbolism to calculate the gradient of $y=x^n$ at x=1, then "curve-sketching", including curves such as $y=|x|$, $y=2^x$, $y=|x^2-1|$, $y=2^{abs\,x}$, using GRADIENT, then $y=|x^2-1|$ at x=-1 using MAGNIFY.

The "gradient investigation" was carried out in session 4 and the computer was available. Two students used GRADIENT, the first to "zoom" in on the origin for sqr(abs(x)) and the second to check that the gradient of y=x (to the left of the origin) was the same as $y=x+x^2$ to the right. After the gradient investigation, SUPERZOOM was used for curve-sketching of rational functions, superimposing f(x) and 1/f(x). The curve-sketching continued throughout session 5.

In session 6, the students did a class investigation into the gradient function of $y=x^2$, then for $y=2^x$ and $y=3^x$ . This was followed by differentiating $y=x^2$ and $y=x^2+3x$ from first principles which the teacher found "took longer than expected".

Session 7 began with the derivative considered as a "measure of the rate of increase of f" and continued with differentiation exercises in which the students calculated f′(x), then checked using GRADIENT in small groups. Finally the teacher looked at $f(x)=x^n$ with n=−1 over the ranges x=−10 to 10, y=−10 to 10 with c=0.001, c=1. The teacher commented "the students may have felt that I was making a mountain out of a molehill in this exercise". Even so, the next lesson continued the same way, with students calculating derivatives and checking the results on th computer. They were asked to suggest a function such that (f(x+c)−f(x))/c did *not* give a curve similar to f′(x) for larger values of c. They were unable to make sugestions, so the teacher demonstrated $y=x^n$ with n=−1 and c=1,2. They also looked at the chord gradients of $y=x^{-3}$, $y=x^2$, $y=x^3$ on the computer using different values of c, including 1, 2, −2, 0.0001. The case of $x^n$ for n=1/3 caused a class discussion on the meaning of $x^{1/3}$ for negative x and they calculated values on the computer and on their calculators. Some calculators could cope, but the computer couldn't; the teacher showed sgnx(absx)$^{1/3}$ to obtain a complete curve for $x^{1/3}$ on the computer.

In session 8 they looked at tangents and normals using standard examples from Bostock & Chandler p.111 example 2 and p.112 14-17. Then $y=x^2-3x+2$ was drawn using SUPERZOOM and various tangents added using the tangent option. The teacher explained how the computer did this as a numerical approximation, then considered the case of abs($x^2-1$) at x=1, drawing lines from x=1 to x=1+1/1000 and x=1 to x=1-1/1000. Then the cursor was moved to zoom in on x=1 to reveal the curve magnifying to two half-lines.

The next day, in session 10 the students did the "tangent investigaton", followed by a discussion of stationary points, with particular emphasis on the case $f'(a)=0$, $f''(a)=0$ might give a maximum, a minimum, or a point of inflexion, by sketching $f(x)=-x^4+1$, $f(x)=x^4+1$ and $f(x)=x^3+1$ on the computer. The students then did some work from Bostock and Chandler (page 122, exercises 1-4) with the computer available for checking results.

In the final session 11, revision of the factorization of cubic expressions was followed by further exercises on maxima and minima. The students were then asked to sketch y=sinx and to use this to draw the graph of y=abs(sinx), which the teacher demonstrated on the computer. This has minima at 0,$\pi$, 2$\pi$ etc, but no derivative there.

The post-test followed in the next session.

## Barton Peverill Experimental Group 2

This experimental group also experienced initial difficulties with the computer. In the first lesson, after the pre-test, only $f(x)=x^2$ was demonstrated. In the second lesson the teacher's diary reported "TV monitor playing up, second computer faulty - wasted valuable time. Spent a long time using computer to calculate various chord gradients. Used students to feed in various data - the data coming from the rest of the students - e.g. values for a & b, or what curve to try. They didn't always get the same results as each other when looking at curves like y=absx and $y=2^{absx}$. ... Homework - think of some interesting curves to try out on the computer."

The third lesson had the students split into six groups of about three, carrying out chord limit exercises from the book and checking the results on the computer. At the end the teacher demonstrated the blancmange program, coping well as the graph was built up, but experiencing difficulties with the magnifying option.

In session 4 the students performed the gradient investigation; the teacher was "determined this investigation should not be done as a class, but in their six groups". They took it in turns whilst the rest "revised algebra". The teacher reported:"

students unfamiliar with the computer seem to be getting more confident", but "one student - very anti computers - becoming very vocal in his comments. He does not like this 'find out yourself' - believes I should tell him all the answers and he'll remember/learn them. He is a very bright student who likes only hard facts - anything vague unnerves him. His efforts in fact bring out the very need for this type of 'discovery' 'thought-provoking' work in class."

Session 5 was used to finish the investigation, with the computer "in constant use and a great deal of discussion went on within, and between, groups. It took a while to accept that some graphs couldn't be put on the computer as they were" (for example graphs with different formula in different parts of the domain). "Couldn't stop discussion, but stressed that they only write down <u>their</u> conclusions."

In session 6 the teacher demonstrated the gradient functions of $y=x^2$, $y=x(2x-1)$ and $y=x^3+1$, following the examples in the text, then the students worked through the exercise at their own speed, as the teacher put "one or two onscreen". "Then those that finished first put rest on computer."

Session 7 began by checking that the students had finished the previous work, then moved on to using the computer for $f(x)=x^n$. "Students fascinated by $y=x^{-1}$ and suggested various values of c to try - giving at first small values of c & then larger ones to watch the 'hiccup'. Discussed the need for limits, but I think one or two thought I was 'pulling a fast one'. Exercise 5C completed with those finishing first using the computer."

Session 8 began with the teacher revising the general cases, differentiation $y=ax^n$ and other polynomials, then carrying out exercises 5D in the text, finishing off for homework as necessary. "Again, as students finished the 3 sections of this exercise, they tried the odd one on the computer - especially Q.32 as answer in the back of the book is wrong!"

Session 9 began with an introduction to tangents and normals, with demonstrations using SUPERZOOM to draw the numerical "tangent" through the points on the graph with x-coordinates x and x+0.00001. The graphs $y=x^2-3x+2$ and $y=abs(x^2-1)$ were drawn by the teacher. "When asked for suggestions for other curves, some said they'd had enough of 'abs' type ones and asked to try y=tanx and were concerned about curves like $y^2=4ax$." (The latter couldn't be drawn by SUPERZOOM, but was in the A-level syllabus.)

In session 10 I visited the school and was asked to talk to the students, particularly the one who was "anti-computer", about the need for this kind of investigation. He was quite firm in his desire to be told the theory and I asked him what he would do if he met a problem of a kind he had not seen before. As an example I asked him what he thought the equation "dy/dx=-y/x" meant. Although he had not met differential equations he asserted that the graph would have a gradient equal to $-y/x$ at any point. Questions to the class of the kind "if the graph went through $x=1, y=2$, what would its gradient be there?" were quickly answered successfully. But no-one was able to suggest what the graph of the solution might be.

At this stage I loaded the first order differential equation program "1stODE" from Graphic Calculus and drew the direction diagram for the equation. We discussed its meaning, then I asked the student what shape a solution might be. He replied "any circle centre the origin". When it was pointed out that he had used the computer to gain insight into a problem he hadn't been able to do before, he commented, with seemingly less conviction, that there must be a rule to answer the problem. I responded that such a rule did exist in this case, but there were some, extremely simple looking, differential equations that had no "rule" to find a formula for the solution, for which numerical methods were the only method of approach. We looked at the differential equation $dy/dx=x^2+y^2$ as an example.

Session 11 continued the classwork with students doing exercise 5E on finding tangents. Two computers were available for student use. The teacher remarked "Mark Lawrence, after all the grumbling and 'anti-computer' complaints, hogged the computer..."

There was some concern in one question that the computer "drew a lovely tangent, but the normal didn't look right as it didn't appear to be at right angles to the tangent... believe it was probably caused by the scaling."

The sheet on "the idea of a tangent" was carried out at the beginning of session 12 (the teacher's report does not mention the use of the computer by the students here). The rest of the lesson was devoted to the idea of stationary values (which the students had met before) and exercises were set with the computer program GRADIENT available for use. The teacher remarked that she "noticed students repeating shortening chord for great length of time until it finally reached a limit to an incredible number of decimal places - it seemed to hypnotise/fascinate them".

Session 13 considered turning points - demonstrated on the computer using GRADIENT, with the computer available to students as they carried out exercises. "Students loved watching the computer plotting the gradient function ... lively discussion on points of inflection ..."

The post-test was taken at the beginning of session 14.

At the end the teacher wrote "I've stuck as close to your instructions as I could. It's taken longer than I intended - not the fault of your idea but of the physical use of the computer - time wasted fetching and returning - only one computer amongst 17. Would have been far more successful in room full of computers with easy access. For most of these students chapter 5 has been revision but has shown up the gaps in their understanding of basics. The students have certainly seen how invaluable the computer is in demonstrating what is happening so clearly (provided they were near enough!)."

## Comparison with the Control Groups

The four other groups at Barton Peverill also kept logs of the work done in class. In outline they followed chapter five of Bostock and Chandler, covering the bookwork as given, with the usual format of teacher explanation, followed by students doing exercises.

The time spent on the material varied widely, from the group BCF (taking further mathematics) who were recorded as having only five sessions on the work (not including the four tests), to the group BC1, who spent 12 sessions (including the tests). Groups BC2 and BC3 spent 10 and 8 sessions respectively, including the tests. By contrast, the experimental groups at Barton Peverill, BE1, BE2 spent 12 and 14 sessions. However, two of the 12 sessions in BE1 were taken up with curve-sketching, reducing the number to a comparable 10 on chords and tangents, whilst one of BE2 was taken up by my visit and others were hampered by difficulties with the computer.

In Kenilworth, where most of the students were taking the subject for the first time, the work on the calculus in the experimental group took eleven sessions, including the tests. The other group was not recorded in detail, however, it took longer covering the material done in the first term as a whole by the experimental group.

Concentrating on the recorded Barton Peverill evidence, it is clear that the use of the computer took longer to cover the same material. Some of this was due to the newness of the computer to the teacher and physical difficulties with hardware. But the main factors were the time taken covering a wider class of examples and non-examples, the time taken for discussion, and the time taken by students taking it in turn to type in their own examples.

Cricklade College

The third institution agreeing to try out the programs was a College of Further Education which acted both as a sixth-form college and as a college for students of a wider range of interests and abilities. Here a junior teacher offered to try out the programs and this was taken up by the Head of Mathematics who agreed to use the computer within their standard format of lectures and examples classes. The College had better computing facilities than either of the other institutions involved, with a room of BBC computers with disc-drives. However, at the outset the Head of Mathematics expressed some scepticism as to the value of computers in mathematics learning.

His report of the activities consisted of a single sheet outlining one session of ninety minutes. This was a "lecture on differentiation from 1st principles (to include the limiting process) ≏ 40 minutes in lecture theatre". It was followed by a "class exercise from B&C P.102 Ex5a" with "no official homework".

There was no use of the computer in the lecture but there was a "demonstration in class tutorial using $y=x^2$ (as directed)" with "groups of 2 or 3 checking answers from the exercise - or doing them for the first time (if early in the rota)".

The comments given afterwards were:

> Plan adhered to. However, we found that students had very mixed reactions to using the computer. The 'Computer Science' students enjoyed it as a piece of computing, appreciating the level of sophistication of the programs and even commenting on ways that they could be improved! A large number did not seem to find using the computer either interesting or useful and much preferred to use paper and pencil to solve the problems. A small number were positively "anti" the computer.
>
> It seems obvious that computers do not switch everyone on. Even where the students found the computer useful, they soon tired of using it to do the exercise. Perhaps there was not enough variety in the exercise - but how could it be any

different?

Not a huge success!

In an accompanying letter the Head of Mathematics wrote

The comments on the lesson plan are, I am sure, somewhat disappointing to you. It seems certain that most students are less interested in understanding calculus than in actually performing well in homeworks and tests i.e. using the rules! Those students who are interested seem to understand the theory well enough in its abstract form.

... To make you feel a bit better, my staff are very impressed indeed with the graph-drawing programs (particularly Superzoom) and have used them in other lessons - particularly with TEC2 maths. Here the students have been drawing graphs in the usual way (pencil and paper and brain), then checking with the computer. This has been much more successful. Perhaps this is because it is not trying to illustrate theory, just performing a standard operation (plotting points).

I am sorry not to have been more help. It has left me rather sceptical about the role of the computer in teaching mathematics. It would seem that it is only useful in straight-forward "calculation" problems - most of which can be done on a calculator...

## Student opinions

The post tests included the opportunity for students who had been exposed to the programs to express opinions about their use. The responses give valuable insight into the student views and help to explain differences between the experimental groups.

The first attitudinal question requested a response on a seven point scale as to whether the programs were helpful or unhelpful (Table 8.3).

<u>Were the programs helpful?</u>

| | very helpful | helpful | fairly helpful | neutral | unhelpful | fairly unhelpful | very unhelpful | No Response |
|---|---|---|---|---|---|---|---|---|
| Kenilworth | 5 | 2 | 6 | 2 | 0 | 0 | 0 | 0 |
| Barton P.1 | 0 | 2 | 7 | 3 | 0 | 0 | 0 | 0 |
| Barton P.2 | 1 | 6 | 8 | 1 | 0 | 0 | 0 | 0 |
| Cricklade | 1 | 2 | 10 | 11 | 5 | 7 | 1 | 14 |
| Total | 7 | 12 | 31 | 17 | 5 | 7 | 1 | 14 |

<u>Table 8.3</u>

The Cricklade students were clearly divided in their opinions, with 13 out of 37 who responded grading the programs in the three "unhelpful" categories as opposed to 11 "neutral" and 13 "helpful". By contrast the three other experimental groups yielded no "unhelpful" responses, 6 "neutral" and 37 "helpful". The responses tended to avoid the extremes of "very helpful" and "very unhelpful", apart from five Kenilworth students (a third of the class) who responded in the "very helpful" category. As this was the class taken by the researcher, the response could be due to a "Hawthorne" effect, or, in part, to the researcher's additional knowledge of the programs and ways of using them. As we shall see shortly, none of the Kenilworth students mentioned a lack of understanding of how the programs worked but this criticism was voiced by some individuals in all the other groups.

The students were asked how many times they had used the programs themselves, in a group of three or less (Table 8.4) It is apparent that several Kenilworth students had used the programs less often than the researcher had hoped. One student, the only girl in the group, had not touched the computer at all. In the two Barton Peverill groups, virtually all students had used the programs at least three or four times whilst the Cricklade students had at most a single opportunity.

How many times did the students use the computer in a small group?

| | 0 | 1 | 2 | 3 | 4 | more than 4 | No Response |
|---|---|---|---|---|---|---|---|
| Kenilworth | 1 | 4 | 1 | 2 | 4 | 3 | 0 |
| Barton P.1 | 0 | 0 | 0 | 1 | 3 | 8 | 0 |
| Barton P.2 | 0 | 0 | 1 | 5 | 0 | 10 | 0 |
| Cricklade | 9 | 26 | 2 | 0 | 0 | 0 | 14 |

Table 8.4

Their opinion was sought as to "how much time did the class as a whole spend using the computer?". (Table 8.5) The responses tallied very much with the previous table. The majority of those from Kenilworth and Barton Peverill saying it was "about right", with half those in the second Barton Peverill group suggesting it was "too much", whereas the majority of the Cricklade students felt the time was "too little" or "far too little".

How much was the computer used by the whole class?

| | far too much | too much | about right | too little | far too little | No Response |
|---|---|---|---|---|---|---|
| Kenilworth | 0 | 2 | 11 | 2 | 0 | 0 |
| Barton P.1 | 1 | 3 | 8 | 0 | 0 | 0 |
| Barton P.2 | 0 | 8 | 8 | 0 | 0 | 0 |
| Cricklade | 1 | 0 | 11 | 16 | 9 | 14 |

Table 8.5

When the students were asked "how much timd did you have using the computer by yourself or with a small group" the responses all skewed further over to the "too little" categories, following the trends found in the previous tables. The Kenilworth students showed most students thought it "about right", with six out of fourteen reporting it to be "too little". The response of the first Barton Peverill group was much the same, with the second group having thirteen out of sixteen thinking it "about right". On the other hand, three quarters of the Cricklade responses considered the time "too little" or "far too little".

How much was the computer used individually or in small groups?

| | far too much | too much | about right | too little | far too little | No Response |
|---|---|---|---|---|---|---|
| Kenilworth | 0 | 0 | 8 | 6 | 0 | 0 |
| Barton P.1 | 0 | 1 | 7 | 4 | 0 | 0 |
| Barton P.2 | 0 | 1 | 13 | 2 | 0 | 0 |
| Cricklade | 0 | 1 | 8 | 10 | 18 | 13 |

Table 8.6

The overall pattern that emerges shows that the experimental groups in Kenilworth and Barton Peverill felt the programs were (fairly) helpful. About the right time was given to class and individual use, with a tendency for too much class time and not enough for individual work.

The majority of Cricklade students, on the other hand, considered that insufficient time was given for the use of the programs, with far too little time for individual use. This difference in exposure is clearly a contributing factor to the Cricklade students' opinion of the helpfulness of the programs.

Helpful and unhelpful aspects

The final part of the questionnaire asked the students to say "in what ways did the computer help?" and "in what ways was the computer unhelpful?". Some students commented only on helpful aspects, some on both sides of the coin and some only on unhelpful aspects. These could be classified as in table 8.7.

Comments on the helpfulness or unhelpfulness of the programs

| | only helpful | helpful & unhelpful | only unhelpful | no comment |
|---|---|---|---|---|
| Kenilworth | 8 | 5 | 0 | 1 |
| Barton P.1 | 3 | 7 | 1 | 1 |
| Barton P.2 | 5 | 11 | 0 | 0 |
| Cricklade | 6 | 7 | 12 | 26 |

Table 8.7

By looking in detail at the responses it is possible to suggest reasons why the groups responded so differently to the earlier questions.

Negative attitudes

The Cricklade teacher reported a number of students "anti-computer". Twelve students commented only on "unhelpful" aspects, but this must be contrasted with the fact that only one student thought the computer was used "too much" in small groups and 28 thought it was used "too little" or "far too little". An analysis of the "unhelpful" comments reveals that nine out of twelve mention lack of time, confusion as to the aims of the program, or that it "kept going wrong". We shall look into these shortly.

Of the remaining three, two restricted their comments to "none" and "none really ... I don't know" whilst the other remarked:

> It did not show me any more than I already knew. (CE50)

Amongst those expressing both positive and negative aspects was the following:

> The computer was alright, but the time it took to show the whole class a simple fact was so long that it wasn't really worth it. (CE43*)

A far more hard line attitude came from one of the Barton Peverill students:

> It did not [help]. The teacher could have easily drawn the graphs it did. When talking about the computer it is made out to be a great deal, but is little more than a tool. It seems pointless learning on one if you can't use it in the exam. You should learn to work things out by yourself rather than on a computer. (BE112*)

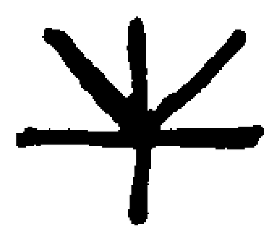
One should remark that there were students who expressed negative attitudes in class (as mentioned in the earlier diaries of events), but these students gave balanced views of the pros and cons which will be reported later.

Confusing features

Only one Kenilworth student (who found the program "very helpful") mentioned a (temporary) confusion:

I got confused when (in calculating gradients) the chord through two very close points was extended - I thought that the chord was a tangent and as a result got confused about the tangent of y=absx at x=0. (KE11)

Five students in the first Barton Peverill group and one in the second mentioned confusing elements:

The graphs shown were often just put in without me understanding what it was doing. e.g. to draw the gradient function or why it couldn't be defined on graphs such as  (BE105*)

Sometimes it was confusing because there was a lot going on at once. (BE107*)

... Got confusing at times. (BE108*)

Some of the functions had to be typed in a special way, which was confusing - we could also have had more explanation as to <u>why</u> this was so. (BE110*)

Sometimes we were confused by exactly what it was doing. It needed more explanation from the teacher about the program. (BE113)

Didn't always explain what it was doing and just put up the answer or solution to the problem. (BE206)

More Cricklade students commented on the lack of explanation:

Difficult to understand the working out. (CE19)

Didn't explain in depth the ideas behind the theory. (CE22*)

There was no explanation of what was happening. (CE23)

It did not give a satisfactory explanation of what it was doing. All we could see was the production of tangents. (CE24)

No explanation - just worked it out - confused me. (CE26)

No specification as to what the aim was. (CE37)

It didn't [help]. Hard to understand. No aim given. (CE38*)

These remarks could be attributed in part to the minimal explanations given in the programs themselves (a limitation exacerbated by limited memory space in the microcomputer). However, the programs were designed for initial teacher demonstration and some student comments suggest that more explanation from the teacher might have been valuable.

## Technical and Organisational factors

At Barton Peverill two students remarked on organisational difficulties with the computer already noted by the teachers:

> We had to go and get it from down the corridor. (BE103*)

> ... much time was taken fetching the computer, setting it up and finding one which worked (several times it didn't). (BE210*)

Cricklade students mentioned difficulties actually getting the program to work:

> It was confusing when it went wrong. (CE02 & CE06)

> It kept going wrong. (CE27)

> Mr **** couldn't remember how to load it. (CE44*)

Difficulties in understanding the precise limitations in typing the formulae for functions have already been mentioned in the previous section. One articulate student (who found the program "very helpful") mentioned his difficulties with the menu and the keyboard:

> The programs themselves were a little difficult to manipulate for a duffer like me with no knowledge or experience of computers. I, for example found it hard to discover what you had to do to draw a tangent or get the cursor into operation. Not knowing where the keys were was a problem. (KE07*)

One computer science student at Kenilworth criticized the program features, partly for lacking symbolic differentiation and partly for a difficulty with the menu:

> It could not derive a function. Poor menus e.g. confusion between draw gradient and draw derivatives. (KE08)

The Time factor

A major criticism at Cricklade was the lack of time allocated:

> Not enough time to use it. (CE04)

> We didn't have enough time to really understand what the computer was showing us; it was rather confusing until you got used to the idea. (CE14*)

> Not enough time to yourself. (CE15)

> It did not [help] because of lack of time. It did not leave

any impression. (CE16)

We had limited time, so not much achieved. More time could mean a better understanding. (CE44*)

Given a time allocation that seemed more appropriate to students in the other experimental classes, the computer programs were seen to speed up graph-sketching, though a minority felt they were holding up progress:

It speeded things up [in sketching graphs]. It kept us from continuing [with work]. (KE15*)

Sometimes showed you how to draw a graph if you weren't sure of it. It wasted time sometimes in just showing us what we had done was right when we already knew it was! (BE104*)

Helped in understanding absolute points and looking at e.g. Q.6. [sketching the derivative of a graph] from a 'pictorial' point of view, rather than from an algebraic point of view. It provided another perspective. Excessive use of it whilst learning things slowed us down ... (BE210*)

It made it easier to picture the graphs and the gradient functions and actually show what differentiation did to graphs. It made it confusing when trying to solve problems numerically, but it was also drawn out (boring). Too much time was spent on the first part (deriving a graph from first principles). (BE214*)

It made it easier to picture the graphs mentally and to understand gradient functions. It took too much time and could not always superimpose the right number of graphs. (BE215*)

[It was helpful] being able to see functions quite quickly. It made the subject boring and confusing. Too much time was spent doing questions using first principles. (BE216*)

Two of the above students (KE15* and BE210*) were those who expressed strong reservations about using computers in class.

Helpful factors

Twelve out of fifty one Cricklade students commented on helpful facets of the program, including the following:

Helped you to see how the gradient of the tangent approached the gradient of the curve as $\delta y/\delta x \to 0$. (CE14*)

It showed you how to work out the answer and sketch diagrams of the gradient. (CE15)

It helped show how a chord on the graph tends to the gradient as the difference in x decreased. (CE20)

It gave an easy to understand diagram and showed all the happenings. (CE23)

Quicker, easier. (CE50)

Some of the "helpful" comments intimated possible misunderstandings:

It showed graphically the idea of tangent tending to gradient. (CE09)

It showed the process by which the gradient of a point could be found clearly. (CE21)

Thirteen out of fourteen Kenilworth students, ten out of twelve from the first Barton Peverill group, and all sixteen in the second group reported ways in which the computer was helpful. A selection of comments are as follows:

It was able to picture and therefore show more clearly the chord of a curve tending to the tangent which a text book would not be able to convey so easily. (KE01)

It enabled me to see the curves I could not visualise. Enabled me to see what was happening to a gradient of a curve as $\delta x \rightarrow 0$. (KE02)

Helped to show how derivatives are obtained from graphs of various functions. (KE03)

The computer helped to make the theory of differentiation, with changes in $\delta x$ and $\delta y$, much simpler. (KE04)

It helped to show how to find the graph of the derivative from the graph of the function itself. (KE06)

It helped to obtain an accurate picture of graphs which even the textbook didn't provide. It saved a lot of time by being able to draw instantly something which would have take far longer to draw less accurately manually. (KE07*)

It gave a good graphical representation of what was happening. "Moving" graphs could be seen more easily i.e. when two points move closer together. (KE08)

It helped to visualise tangents, gradients derivatives etc, in a graphical [way] instead of purely algebraically. It also made the lesson much more interesting! (KE11)

As a way of seeing the true nature of the graphs, especially in magnification, and to see the actual process of differentiation from first principles. (KE16*)

It could show chords being drawn which get nearer to a tangent so that you could see the gradient getting almost the same at each point close to the one being investigated. (BE105*)

It gave a clear indication of how the graph was drawn and what it looked like. (BE106*)

It helped in giving a clear picture of the graphs of some functions, which were otherwise difficult to draw or understand, and also helped in picturing the gradient functions of those graphs. (BE110*)

It showed us exact graphs & gradient functions whereas we could only draw it roughly. (BE113*)

Showed more of the things in a graphical manner which was easy to understand. (BE202*)

Showing gradient function - plotting points helped to understand it. (BE203*)

It helped when we needed a pictorial view of what we were doing, something which I couldn't imagine. It helped us to answer questions. I can't think of any unhelpful thing. (BE207*)

You could picture the curves and see their shape, giving a very good visual image. (BE209*)

Helped me to appreciate what it was I was being taught and how the formulae were derived. (BE213*)

It made it easier to picture the graphs mentally and to understand gradient functions. (BE214*)

**Comment**

Whilst there were minor flaws in presentation of the material at both Kenilworth and Barton Peverill, thirty nine out of forty two students reported ways in which the programs were helpful. When asked they reported technical difficulties and lack of explanation at times, but this is in the overall context that none of them regarded the programs as "unhelpful" (table 8.3).

At Cricklade most students commented that they did not have enough time using the programs (tables 8.5 and 8.6), and there were individual comments reporting confusion due to lack of time and explanation.

In the analysis of pre- and post-tests which follows in the next chapter, the Cricklade results will be considered separately from the others.

## 9. Analysis of Responses: Mathematical Processes

The classwork was preceded by a short pre-test, which was repeated afterwards as a post-test with additional questions on the work that had been covered. In this chapter we shall consider those questions in which the students were asked to carry out various mathematical processes:

(A) Calculating numerical gradients from a picture,

(B) A simple example of differentiation from first principles,

(C) Formal differentiation of polynomials and powers,

(D) Sketching the derivative for a given graph,

(E) Recognizing a derivative,

(F) Specifying a non-differentiable function.

It may be expected that the use of the computer and the teaching and investigations that accompanied it would lead to a significant improvement in (D), (E) and (F), with a possible improvement in (A), but there is no reason to expect any improvement in (B) and (C).

In the following descriptions the symbols KE, BE1, BE2 refer to the Kenilworth and Barton Peverill experimental groups, CE to Cricklade and KC, BC1 to BC4 the Kenilworth and Barton Peverill controls. U denotes the group of university students (at least one grade A in A-level mathematics) in their first week at

Warwick University. Students who missed the pre- or post-test in the sixth-form groups were eliminated from the statistics. The numbers of students in each group responding both times, and the numbers, in brackets, who had previous experience of calculus are as in table 9.1:

| | |
|---|---|
| KE | 14 (3) |
| BE1 | 12(12) |
| BE2 | 16(15) |
| CE | 51(14) |
| KC | 9 (2) |
| BC1 | 15(10) |
| BC2 | 18(18) |
| BC3 | 14(11) |
| BC4 | 11(11) |
| U | 44(44) |

Table 9.1

Students in each group will be numbered so that, for example, BC205 is the fifth member of group BC2; those with previous calculus experience will also be marked with an asterisk.

The performance of the groups on the relevant questions in the pretest and posttest are given in Tables 9.2 to 9.10 and will be analysed in the rest of this chapter. A cursory glance shows a significantly better performance by the experimental groups on the posttest in the tasks (D), (E), (F) covered by questions 6, 7, 8 respectively. The experimental students clearly get more marks on question 6 and the notes following each table show they obtain more correct responses in questions 7, 8.

Kenilworth Experimental Group

No. of students:14

| question | 1pre/post | 5pre/post | 6a | 6b | 6c | 6d | total | Q7 | Q8 | comments on Q6 (if any) |
|---|---|---|---|---|---|---|---|---|---|---|
| max. | 12/12 | 12/12 | 5 | 5 | 5 | 5 | 20 | | | |
| KE01 | 11/11 | -/10 | 5 | 4 | 5 | 5 | 19 | 4s | - | |
| KE02 | 10/11 | -/7 | 5 | 5 | 5 | 5 | 20 | 2c | fg | |
| KE03 | 10/11 | -/8 | 5 | 0 | 0 | 0 | 5 | 3i | - | |
| KE05 | 9/12 | -/10 | 5 | 5 | 5 | 4 | 19 | 2c | if | |
| KE06 | 12/12 | -/12 | 5 | 5 | 5 | 5 | 20 | 2c | f | |
| KE07* | 12/12 | 8/10 | 5 | 5 | 5 | 5 | 20 | 2c | if | |
| KE08 | 12/11 | -/7 | 5 | 5 | 5 | 5 | 20 | 2c | f | |
| KE09 | 10/11 | -/4 | 4 | 5 | 3 | 0 | 12 | 2- | if | |
| KE11 | 11/12 | -/12 | 5 | 5 | 5 | 4 | 19 | 2c | f | |
| KE12 | 10/12 | -/10 | 5 | 5 | 5 | 5 | 20 | 3i | e | |
| KE13 | 12/12 | -/12 | 5 | 5 | 5 | 5 | 20 | 2c | f | |
| KE14 | 6/6 | -/12 | 5 | 5 | 3 | 4 | 17 | 2c | f | |
| KE15* | 10/12 | 8/12 | 5 | 4 | 5 | 5 | 19 | 2c | if | |
| KE16* | 12/12 | 9/12 | 5 | 5 | 5 | 5 | 20 | 2c | if | |
| mean | 10.50/11.21 | 8.33/ 9.86 | 4.93 | 4.50 | 4.36 | 4.07 | 17.86 | | | |
| S.D. | 1.59/ 1.52 | 0.47/ 2.42 | 0.26 | 1.30 | 1.39 | 1.71 | 4.12 | | | |

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NOTES:
* following a student's name indicates previous experience of calculus

Question 1: Calculating numerical gradients
Question 5: Formulae for differentiation
Question 6: Sketching the derivative for a given graph

---

Question 7 (recognising a derivative)
Of the three choices, 2 is correct, 3 & 4 are incorrect, with 4 having graph & derivative of similar shape.
Codes are c:correct, i:incorrect, s:'same shape' (incorrect), ?:researcher unsure, -:no explanation.

| response: | 2 | 3 | 4 | nr |
|---|---|---|---|---|
| | 11 | 2 | 1 | 0 |

correct responses with reason................10 out of 14 (71%)

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Question 8 (specifying a non-differentiable function)

Codes are :f:formula given, g:graph drawn, e:written explanation only (all considered satisfactory)
if:incorrect formula, ig:incorrect graph, ie:incorrect explanation, -:no response (all unsatisfactory)

Satisfactory responses......................7 out of 14 (50%)

Table 9.2

Barton Peverill Experimental Group 1

No. of students:12

| question | 1pre/post | 5pre/post | 6a | 6b | 6c | 6d | total | Q7 | Q8 | comments on Q6 (if any) |
|---|---|---|---|---|---|---|---|---|---|---|
| max. | 12/12 | 12/12 | 5 | 5 | 5 | 5 | 20 | | | |
| BE101* | 9/5 | 12/12 | 5 | 1 | 0 | 1 | 7 | 4s | g | three graph shapes |
| BE102* | 11/11 | 12/12 | 5 | 5 | 5 | 5 | 20 | 2c | f | |
| BE103* | 11/11 | 12/12 | 5 | 5 | 5 | 5 | 20 | 2c | f | |
| BE104* | 0/8 | 12/12 | 5 | 2 | 2 | 0 | 9 | 3i | if | one graph like, one mirror image |
| BE105* | 12/12 | 7/12 | 5 | 5 | 5 | 5 | 20 | 2c | g | |
| BE106* | 11/10 | 12/12 | 5 | 3 | 0 | 0 | 8 | 3i | - | |
| BE107* | 12/11 | 7/10 | 5 | 5 | 5 | 4 | 19 | 2? | f | |
| BE108* | 10/11 | 9/10 | 5 | 5 | 5 | 0 | 15 | 3i | if | |
| BE109* | 6/12 | 12/12 | 5 | 1 | 0 | 1 | 7 | 4s | - | three graph like |
| BE110* | 11/12 | 12/12 | 5 | 5 | 5 | 5 | 20 | 2c | if | |
| BE112* | 10/11 | 12/10 | 5 | 5 | 5 | 3 | 18 | 2c | g | |
| BE113* | 11/12 | 12/12 | 5 | 5 | 5 | 5 | 20 | 2c | if | |
| mean | 9.50/10.50 | 10.92/11.50 | 5.00 | 3.92 | 3.50 | 2.83 | 15.25 | | | |
| S.D. | 3.25/ 1.98 | 1.93/ 0.87 | 0.00 | 1.61 | 2.18 | 2.15 | 5.49 | | | |

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NOTES:
* following a student's name indicates previous experience of calculus

Question 1: Calculating numerical gradients
Question 5: Formulae for differentiation
Question 6: Sketching the derivative for a given graph

---

Question 7 (recognising a derivative)
Of the three choices, 2 is correct, 3 & 4 are incorrect, with 4 having graph & derivative of similar shape.
Codes are c:correct, i:incorrect, s:'same shape' (incorrect), ?:researcher unsure, -:no explanation.

| response: | 2 | 3 | 4 | nr |
|---|---|---|---|---|
| | 7 | 3 | 2 | 0 |

correct responses with reason................6 out of 12 (50%)

---

Question 8 (specifying a non-differentiable function)

Codes are :f:formula given, g:graph drawn, e:written explanation only (all considered satisfactory)
if:incorrect formula, ig:incorrect graph, ie:incorrect explanation, -:no response (all unsatisfactory)

Satisfactory responses.......................6 out of 12 (50%)

Table 9.3

Barton Peverill Experimental Group 2

No. of students:16

| question | 1pre/post | 5pre/post | 6a | 6b | 6c | 6d | total | Q7 | Q8 | comments on Q6 (if any) |
|---|---|---|---|---|---|---|---|---|---|---|
| max. | 12/12 | 12/12 | 5 | 5 | 5 | 5 | 20 | | | |
| BE201* | 12/12 | 10/12 | 5 | 5 | 5 | 5 | 20 | 2c | if+ | |
| BE202* | 10/12 | 6/8 | 5 | 3 | 5 | 5 | 18 | 2c | if+ | |
| BE203* | 9/11 | 12/12 | 5 | 5 | 5 | 3 | 18 | 2c | if+ | |
| BE204* | 11/11 | 6/12 | 5 | 5 | 2 | 2 | 14 | 2? | if+ | |
| BE205* | 12/11 | 8/12 | 5 | 4 | 3 | 5 | 17 | 4s | f! | |
| BE206 | 11/11 | -/10 | 0 | 5 | 5 | 2 | 12 | 4s | | |
| BE207* | 6/11 | 4/8 | 5 | 5 | 5 | 4 | 19 | 3i | if | |
| BE208* | 12/12 | 10/12 | 5 | 5 | 5 | 4 | 19 | 2c | if+ | |
| BE209* | 12/12 | 12/12 | 5 | 5 | 5 | 3 | 18 | 2c | if | |
| BE210* | 12/12 | 12/12 | 5 | 3 | 5 | 5 | 18 | 2c | f | |
| BE211* | 6/12 | 0/12 | 5 | 5 | 5 | 5 | 20 | 2c | if+ | |
| BE212* | 12/12 | 12/12 | 5 | 5 | 5 | 5 | 20 | 2c | if+ | |
| BE213* | 12/9 | 12/12 | 5 | 5 | 5 | 5 | 20 | 2c | - | |
| BE214* | 11/11 | 4/12 | 4 | 3 | 5 | 4 | 16 | 2c | - | |
| BE215* | 9/11 | 10/12 | 5 | 2 | 1 | 5 | 13 | 2c | - | |
| BE216* | 12/11 | 8/8 | 5 | 3 | 5 | 1 | 14 | 2c | - | |
| mean | 10.56/11.31 | 8.40/11.12 | 4.62 | 4.25 | 4.44 | 3.94 | 17.25 | | | |
| S.D. | 2.00/ 0.77 | 3.59/ 1.58 | 1.22 | 1.03 | 1.22 | 1.30 | 2.59 | | | |

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NOTES:
* following a student's name indicates previous experience of calculus

Question 1: Calculating numerical gradients
Question 5: Formulae for differentiation
Question 6: Sketching the derivative for a given graph

---

Question 7 (recognising a derivative)
Of the three choices, 2 is correct, 3 & 4 are incorrect, with 4 having graph & derivative of similar shape.
Codes are c:correct, i:incorrect, s:'same shape' (incorrect), ?:researcher unsure, -:no explanation.

| response: | 2 | 3 | 4 | nr |
|---|---|---|---|---|
| | 13 | 1 | 2 | 0 |

correct responses with reason................12 out of 16 (75%)

---

Question 8 (specifying a non-differentiable function)

Codes are :f:formula given, g:graph drawn, e:written explanation only (all considered satisfactory)
if:incorrect formula, ig:incorrect graph, ie:incorrect explanation, -:no response (all unsatisfactory)

Satisfactory responses......................2 out of 16 (13%)

Table 9.4

Kenilworth Control Group

No. of students:9

| question | 1pre/post | 5pre/post | 6a | 6b | 6c | 6d | total | Q7 | Q8 | comments on Q6 (if any) |
|---|---|---|---|---|---|---|---|---|---|---|
| max. | 12/12 | 12/12 | 5 | 5 | 5 | 5 | 20 | | | |
| KC01 | 12/11 | -/6 | 2 | 5 | 0 | 0 | 7 | 2c | - | 2 formulae |
| KC02 | 12/11 | -/12 | 5 | 5 | 0 | 0 | 10 | 3i | if | 2 formulae |
| KC03* | 9/12 | 8/10 | 2 | 0 | 5 | 0 | 7 | 2c | - | 2 graph shape one mirror image |
| KC04 | 11/11 | -/8 | 5 | 5 | 0 | 0 | 10 | 4- | - | 2 formulae |
| KC05* | 11/11 | 4/4 | 5 | 5 | 1 | 0 | 11 | 4- | - | 2 formulae |
| KC06 | 12/12 | -/6 | 5 | 5 | 0 | 0 | 10 | - | - | 2 formulae |
| KC07 | 11/11 | -/12 | 5 | 5 | 0 | 0 | 10 | 3- | if | 2 formulae |
| KC08 | 12/12 | -/8 | 2 | 5 | 1 | 0 | 8 | 2c | f | 2 formulae |
| KC09 | 11/12 | -/4 | 2 | 1 | 0 | 0 | 3 | 4- | - | four graph shapes |
| mean | 11.22/11.44 | 6.00/ 7.78 | 3.67 | 4.00 | 0.78 | 0.00 | 8.44 | | | |
| S.D. | 0.92/ 0.50 | 2.00/ 2.90 | 1.49 | 1.89 | 1.55 | 0.00 | 2.36 | | | |

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NOTES:
* following a student's name indicates previous experience of calculus

Question 1: Calculating numerical gradients
Question 5: Formulae for differentiation
Question 6: Sketching the derivative for a given graph

---

Question 7 (recognising a derivative)
Of the three choices, 2 is correct, 3 & 4 are incorrect, with 4 having graph & derivative of similar shape.
Codes are c:correct, i:incorrect, s:'same shape' (incorrect), ?:researcher unsure, -:no explanation.

| response: | 2 | 3 | 4 | nr |
|---|---|---|---|---|
| | 3 | 2 | 3 | 1 |

correct responses with reason................3 out of 9 (33%)

---

Question 8 (specifying a non-differentiable function)

Codes are :f:formula given, g:graph drawn, e:written explanation only (all considered satisfactory)
if:incorrect formula, ig:incorrect graph, ie:incorrect explanation, -:no response (all unsatisfactory)

Satisfactory responses......................1 out of 9 (11%)

Table 9.5

Barton Peverill Control Group 1

No. of students:15

| question | 1pre/post | 5pre/post | 6a | 6b | 6c | 6d | total | Q7 | Q8 | comments on Q6 (if any) |
|---|---|---|---|---|---|---|---|---|---|---|
| max. | 12/12 | 12/12 | 5 | 5 | 5 | 5 | 20 | | | |
| BC101* | 11/11 | 8/12 | 3 | 5 | 0 | 0 | 8 | 4s | if+ | |
| BC102* | 8/8 | 0/8 | 0 | 0 | 0 | 0 | 0 | 4s | - | no attempt |
| BC103* | 6/11 | 4/12 | 5 | 3 | 0 | 0 | 8 | 3- | - | one formula, one single tangent |
| BC104* | 6/10 | 4/12 | 0 | 0 | 0 | 0 | 0 | 4s | - | single tangents |
| BC105 | 10/11 | -/12 | 0 | 0 | 0 | 0 | 0 | - | - | no attempt |
| BC106* | 12/12 | 2/12 | 0 | 0 | 0 | 0 | 0 | 4s | - | four single tangents |
| BC107 | 8/5 | -/10 | 0 | 0 | 0 | 0 | 0 | 4s | if | incorrect formulae |
| BC108* | 6/8 | 4/12 | 0 | 0 | 0 | 0 | 0 | 4s | if | four local attempts |
| BC109* | 10/12 | 4/12 | 5 | 0 | 0 | 0 | 5 | 3- | if | |
| BC110* | 12/11 | 9/12 | 0 | 0 | 0 | 0 | 0 | 3- | if | three single tangents |
| BC111* | 7/11 | 12/12 | 5 | 0 | 0 | 0 | 5 | 3- | - | |
| BC112 | 11/12 | -/10 | 3 | 0 | 0 | 0 | 3 | 4s | - | four graph shapes |
| BC113 | 9/11 | -/6 | 5 | 0 | 0 | 0 | 5 | 3- | - | formula |
| BC114* | 4/9 | 4/5 | 0 | 0 | 0 | 0 | 0 | 4? | - | four single tangents |
| BC115 | 11/11 | -/12 | 3 | 5 | 0 | 3 | 11 | 4- | - | two formulae |
| mean | 8.73/10.20 | 5.10/10.60 | 1.93 | 0.87 | 0.00 | 0.20 | 3.00 | | | |
| S.D. | 2.43/ 1.87 | 3.36/ 2.30 | 2.17 | 1.78 | 0.00 | 0.75 | 3.63 | | | |

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NOTES:
* following a student's name indicates previous experience of calculus

Question 1: Calculating numerical gradients
Question 5: Formulae for differentiation
Question 6: Sketching the derivative for a given graph

---

Question 7 (recognising a derivative)
Of the three choices, 2 is correct, 3 & 4 are incorrect, with 4 having graph & derivative of similar shape.
Codes are c:correct, i:incorrect, s:'same shape' (incorrect), ?:researcher unsure, -:no explanation.

| response: | 2 | 3 | 4 | nr |
|---|---|---|---|---|
| | 0 | 5 | 9 | 1 |

correct responses with reason................0 out of 15 (0%)

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Question 8 (specifying a non-differentiable function)

Codes are :f:formula given, g:graph drawn, e:written explanation only (all considered satisfactory)
if:incorrect formula, ig:incorrect graph, ie:incorrect explanation, -:no response (all unsatisfactory)

Satisfactory responses......................0 out of 15 (0%)

Table 9.6

Barton Peverill Control Group 2

No. of students:18

| question | 1pre/post | 5pre/post | 6a | 6b | 6c | 6d | total | Q7 | Q8 | comments on Q6 (if any) |
|---|---|---|---|---|---|---|---|---|---|---|
| max. | 12/12 | 12/12 | 5 | 5 | 5 | 5 | 20 | | | |
| BC201* | 11/11 | 8/12 | 5 | 5 | 2 | 0 | 12 | 2i | if | one graph shape |
| BC202* | 12/9 | 10/10 | 5 | 3 | 0 | 0 | 8 | 3- | if | |
| BC203* | 12/12 | 0/12 | 0 | 0 | 0 | 0 | 0 | 4- | if | no attempt |
| BC204* | 10/10 | 12/12 | 5 | 3 | 3 | 2 | 13 | 3i | if | two graph shapes |
| BC205* | 11/11 | 11/8 | 5 | 5 | 5 | 3 | 18 | 2i | if | |
| BC206* | 11/10 | 12/12 | 5 | 3 | 0 | 0 | 8 | 4i | if | two formulae |
| BC207* | 12/11 | 12/12 | 5 | 5 | 2 | 3 | 15 | 2c | - | two graph shapes |
| BC208* | 12/10 | 12/12 | 5 | 5 | 0 | 0 | 10 | - | - | |
| BC210* | 12/12 | 6/12 | 5 | 5 | 2 | 0 | 12 | 4i | - | |
| BC211* | 12/12 | 6/12 | 2 | 3 | 0 | 0 | 5 | - | - | one formula |
| BC212* | 11/11 | 10/12 | 5 | 5 | 4 | 0 | 14 | 3i | - | |
| BC213* | 12/8 | 12/10 | 5 | 3 | 0 | 0 | 8 | 4- | if | |
| BC214* | 11/12 | 7/12 | 5 | 5 | 0 | 1 | 11 | - | - | two formulae |
| BC215* | 11/12 | 4/11 | 0 | 0 | 0 | 0 | 0 | 4- | if | no attempt |
| BC216* | 10/11 | 8/12 | 5 | 3 | 5 | 3 | 16 | 4- | - | |
| BC217* | 11/12 | 12/12 | 5 | 5 | 5 | 5 | 20 | 2c? | - | |
| BC218* | 12/12 | 11/12 | 2 | 5 | 0 | 0 | 7 | 2? | - | |
| BC219* | 11/11 | 4/10 | 5 | 5 | 0 | 0 | 10 | 4- | - | |
| mean | 11.33/10.94 | 8.72/11.39 | 4.11 | 3.78 | 1.56 | 0.94 | 10.39 | | | |
| S.D. | 0.67/ 1.13 | 3.48/ 1.11 | 1.73 | 1.62 | 1.95 | 1.51 | 5.28 | | | |

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NOTES:
* following a student's name indicates previous experience of calculus

Question 1: Calculating numerical gradients
Question 5: Formulae for differentiation
Question 6: Sketching the derivative for a given graph

---

Question 7 (recognising a derivative)
Of the three choices, 2 is correct, 3 & 4 are incorrect, with 4 having graph & derivative of similar shape.
Codes are c:correct, i:incorrect, s:'same shape' (incorrect), ?:researcher unsure, -:no explanation.

| response: | 2 | 3 | 4 | nr |
|---|---|---|---|---|
| | 5 | 3 | 7 | 3 |

correct responses with reason................2 out of 18 (11%)

---

Question 8 (specifying a non-differentiable function)

Codes are :f:formula given, g:graph drawn, e:written explanation only (all considered satisfactory)
if:incorrect formula, ig:incorrect graph, ie:incorrect explanation, -:no response (all unsatisfactory)

Satisfactory responses......................0 out of 18 (0%)

Table 9.7

Barton Peverill Control Group 3

No. of students:14

| question | 1pre/post | 5pre/post | 6a | 6b | 6c | 6d | total | Q7 | Q8 | comments on Q6 (if any) |
|---|---|---|---|---|---|---|---|---|---|---|
| max. | 12/12 | 12/12 | 5 | 5 | 5 | 5 | 20 | | | |
| BC302 | 4/11 | -/12 | 0 | 0 | 0 | 0 | 0 | - | - | single tangents |
| BC303* | 12/11 | 12/12 | 5 | 0 | 0 | 0 | 5 | 4i | - | |
| BC304 | 8/12 | -/12 | 0 | 0 | 0 | 0 | 0 | 4s | if | single tangents |
| BC305* | 10/11 | 4/12 | 5 | 3 | 0 | 0 | 8 | 2i | if | two graph shapes |
| BC306* | 6/12 | 9/12 | 0 | 0 | 0 | 0 | 0 | 2c | - | no attempt |
| BC308* | 12/12 | 4/12 | 5 | 0 | 0 | 0 | 5 | - | - | |
| BC309* | 11/11 | 1/12 | 2 | 5 | 0 | 0 | 7 | 2c | - | single tangent |
| BC310* | 8/11 | 6/7 | 5 | 5 | 1 | 1 | 12 | 3i | - | |
| BC311* | 7/9 | 12/11 | 1 | 5 | 0 | 0 | 6 | 2c | if | two formulae, tangent |
| BC313* | 7/11 | 4/12 | 0 | 0 | 0 | 0 | 0 | - | - | no attempt |
| BC314 | 12/12 | -/11 | 2 | 1 | 0 | 0 | 3 | 4- | - | four graph shapes |
| BC315* | 12/12 | 8/12 | 5 | 0 | 0 | 0 | 5 | 3i | - | |
| BC316* | 11/11 | 0/12 | 0 | 0 | 0 | 0 | 0 | 2i? | - | no attempt |
| BC318* | 10/11 | 0/12 | 3 | 5 | 1 | 0 | 9 | 3i | if | |
| mean | 9.29/11.21 | 5.45/11.50 | 2.36 | 1.71 | 0.14 | 0.07 | 4.29 | | | |
| S.D. | 2.52/ 0.77 | 4.16/ 1.30 | 2.16 | 2.22 | 0.35 | 0.26 | 3.79 | | | |

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NOTES:
* following a student's name indicates previous experience of calculus

Question 1: Calculating numerical gradients
Question 5: Formulae for differentiation
Question 6: Sketching the derivative for a given graph

---

Question 7 (recognising a derivative)
Of the three choices, 2 is correct, 3 & 4 are incorrect, with 4 having graph & derivative of similar shape.
Codes are c:correct, i:incorrect, s:'same shape' (incorrect), ?:researcher unsure, -:no explanation.

| response: | 2 | 3 | 4 | nr |
|---|---|---|---|---|
| | 5 | 3 | 3 | 3 |

correct responses with reason................3 out of 14 (21%)

---

Question 8 (specifying a non-differentiable function)

Codes are :f:formula given, g:graph drawn, e:written explanation only (all considered satisfactory)
if:incorrect formula, ig:incorrect graph, ie:incorrect explanation, -:no response (all unsatisfactory)

Satisfactory responses.......................0 out of 14 (0%)

Table 9.8

Barton Peverill Control Group 4

No. of students:11

| question | 1pre/post | 5pre/post | 6a | 6b | 6c | 6d | total | Q7 | Q8 | comments on Q6 (if any) |
|---|---|---|---|---|---|---|---|---|---|---|
| max. | 12/12 | 12/12 | 5 | 5 | 5 | 5 | 20 | | | |
| BC401* | 8/11 | 10/12 | 5 | 5 | 1 | 0 | 11 | 2i | - | two formulae |
| BC402* | 12/11 | 12/12 | 5 | 5 | 5 | 5 | 20 | 2- | - | two formulae & perceptive remark |
| BC403* | 10/12 | 4/12 | 5 | 1 | 0 | 0 | 6 | 4s | - | |
| BC404* | 11/10 | 12/12 | 5 | 3 | 3 | 4 | 15 | 2c | if | three graph shapes |
| BC405* | 10/12 | 12/12 | 5 | 5 | 5 | 0 | 15 | 2? | - | |
| BC406* | 11/11 | 12/12 | 5 | 5 | 5 | 5 | 20 | 2c | - | |
| BC407* | 11/12 | 12/12 | 5 | 5 | 5 | 5 | 20 | 3i | - | one formula |
| BC408* | 11/9 | 12/10 | 5 | 5 | 4 | 4 | 18 | 2c | if | (one formula?) |
| BC409* | 11/12 | 12/12 | 5 | 1 | 3 | 0 | 9 | 2c | - | |
| BC410* | 11/9 | 6/12 | 0 | 0 | 0 | 0 | 0 | 4- | - | no attempt |
| BC412* | 10/11 | 10/12 | 5 | 0 | 0 | 1 | 6 | 3i | if | |
| mean | 10.55/10.91 | 10.36/11.82 | 4.55 | 3.18 | 2.82 | 2.18 | 12.73 | | | |
| S.D. | 0.99/ 1.08 | 2.67/ 0.57 | 1.44 | 2.12 | 2.08 | 2.25 | 6.51 | | | |

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NOTES:
* following a student's name indicates previous experience of calculus

Question 1: Calculating numerical gradients
Question 5: Formulae for differentiation
Question 6: Sketching the derivative for a given graph

---

Question 7 (recognising a derivative)
Of the three choices, 2 is correct, 3 & 4 are incorrect, with 4 having graph & derivative of similar shape.
Codes are c:correct, i:incorrect, s:'same shape' (incorrect), ?:researcher unsure, -:no explanation.

| response: | 2 | 3 | 4 | nr |
|---|---|---|---|---|
| | 7 | 2 | 2 | 0 |

correct responses with reason................4 out of 11 (36%)

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Question 8 (specifying a non-differentiable function)

Codes are :f:formula given, g:graph drawn, e:written explanation only (all considered satisfactory)
if:incorrect formula, ig:incorrect graph, ie:incorrect explanation, -:no response (all unsatisfactory)

Satisfactory responses.......................0 out of 11 (0%)

Table 9.9

Cricklade College

No. of students:51

| question | 1pre/post | 5pre/post | 6a | 6b | 6c | 6d | total | Q7 | Q8 | comments on Q6 (if any) |
|---|---|---|---|---|---|---|---|---|---|---|
| max. | 12/12 | 12/12 | 5 | 5 | 5 | 5 | 20 | | | |
| CE01* | 1/3 | 6/12 | 0 | 0 | 0 | 0 | 0 | 4- | - | no attempt |
| CE02 | 11/11 | -/10 | 0 | 0 | 0 | 0 | 0 | - | - | individual tangents |
| CE03 | 12/12 | -/6 | 0 | 0 | 0 | 0 | 0 | - | - | no attempt |
| CE04 | 9/5 | -/5 | 1 | 1 | 0 | 0 | 2 | - | - | two formulae |
| CE05 | 8/12 | -/5 | 0 | 0 | 0 | 0 | 0 | 2i | if | no attempt |
| CE06 | 10/12 | -/4 | 0 | 0 | 0 | 0 | 0 | - | - | individual tangents |
| CE07 | 2/0 | -/5 | 0 | 0 | 0 | 0 | 0 | 4i | if | individual tangents |
| CE08* | 12/12 | 12/12 | 5 | 0 | 0 | 0 | 5 | 4i | - | |
| CE09 | 6/11 | -/12 | 0 | 0 | 0 | 0 | 0 | - | - | no attempt |
| CE10 | 6/6 | -/12 | 0 | 0 | 0 | 0 | 0 | 2- | - | no attempt |
| CE11 | 9/8 | -/4 | 0 | 0 | 0 | 0 | 0 | - | - | no attempt |
| CE12 | 10/11 | -/12 | 0 | 0 | 0 | 0 | 0 | - | - | attempt at one formula |
| CE13* | 12/11 | 12/11 | 2 | 0 | 0 | 0 | 2 | 4- | if | individual tangent ? |
| CE14* | 7/8 | 6/12 | 5 | 2 | 4 | 1 | 12 | - | - | |
| CE15 | 10/5 | -/9 | 5 | 0 | 0 | 0 | 5 | - | - | odd lines (not tangents) |
| CE16 | 9/11 | -/8 | 0 | 0 | 0 | 0 | 0 | 4- | - | individual tangents |
| CE17 | 12/10 | -/10 | 0 | 0 | 0 | 0 | 0 | 4s | - | individual tangents (?) |
| CE18 | 8/12 | 0/12 | 0 | 0 | 0 | 0 | 0 | 4s | - | no attempt |
| CE19 | 11/10 | -/12 | 0 | 0 | 0 | 0 | 0 | - | - | no attempt |
| CE20 | 8/11 | -/6 | 1 | 0 | 0 | 0 | 1 | 3- | if | four graph shapes |
| CE21 | 11/11 | -/9 | 1 | 0 | 0 | 0 | 1 | 2- | - | |
| CE22* | 1/11 | 5/12 | 0 | 0 | 0 | 0 | 0 | 4- | if | one formula |
| CE23 | 11/12 | -/12 | 1 | 0 | 0 | 0 | 1 | 4i | if | four graph shapes |
| CE24 | 9/12 | -/12 | 2 | 0 | 0 | 0 | 2 | 4s | if | one formula |
| CE25 | 9/9 | -/12 | 0 | 0 | 0 | 0 | 0 | 4- | - | no attempt |
| CE26 | 7/12 | -/12 | 0 | 0 | 0 | 0 | 0 | 4s | - | no attempt |
| CE27 | 0/8 | -/10 | 0 | 0 | 0 | 0 | 0 | - | - | individual tangents |
| CE28 | 10/4 | -/12 | 0 | 0 | 0 | 0 | 0 | 3- | - | no attempt |
| CE29 | 5/8 | -/4 | 0 | 0 | 0 | 0 | 0 | - | - | no attempt |
| CE30 | 10/8 | -/6 | 0 | 0 | 0 | 0 | 0 | - | - | no attempt |
| CE31 | 10/9 | -/10 | 0 | 0 | 0 | 0 | 0 | - | - | no attempt |
| CE32 | 10/8 | -/6 | 0 | 0 | 0 | 0 | 0 | - | - | no attempt |
| CE33 | 11/11 | -/12 | 0 | 0 | 0 | 0 | 0 | - | - | no attempt |
| CE34 | 11/10 | -/12 | 0 | 0 | 0 | 0 | 0 | - | - | no attempt |
| CE35* | 11/11 | 0/- | 0 | 0 | 0 | 0 | 0 | - | - | no attempt |
| CE36* | 10/12 | 10/- | 0 | 0 | 0 | 0 | 0 | - | - | no attempt |
| CE37 | 9/11 | -/12 | 0 | 0 | 0 | 0 | 0 | - | - | no attempt |
| CE38* | 11/11 | 12/12 | 0 | 0 | 0 | 0 | 0 | - | - | no attempt |
| CE39* | 4/8 | 5/9 | 5 | 1 | 0 | 0 | 6 | 4- | - | |
| CE40 | 12/11 | -/11 | 0 | 0 | 0 | 0 | 0 | - | - | no attempt |
| CE41 | 12/12 | -/11 | 5 | 5 | 0 | 0 | 10 | - | - | |
| CE42* | 11/9 | 10/6 | 0 | 0 | 0 | 0 | 0 | - | - | no attempt |
| CE43* | 11/11 | 12/12 | 0 | 0 | 0 | 0 | 0 | 4- | - | two attempted formulae |
| CE44* | 12/8 | 12/12 | 0 | 0 | 0 | 0 | 0 | 4i | if | individual tangents |
| CE45 | 7/8 | -/12 | 0 | 0 | 0 | 0 | 0 | 2i | - | two wrong attempts at formulae |
| CE46 | 9/6 | -/10 | 0 | 0 | 0 | 0 | 0 | 4- | if | individual tangents |
| CE47 | 10/11 | -/8 | 4 | 3 | 0 | 1 | 8 | - | - | |
| CE48 | 10/9 | -/5 | 0 | 0 | 0 | 0 | 0 | 3- | | no attempt |
| CE49 | 0/8 | -/8 | 0 | 0 | 0 | 0 | 0 | 4- | | two wrong attempts at formulae |
| CE50 | 2/8 | -/12 | 0 | 0 | 0 | 0 | 0 | - | - | no attempt |
| CE51* | 10/11 | 12/12 | 0 | 0 | 0 | 0 | 0 | - | - | no attempt |

Total statistics overleaf

Cricklade College: statistics

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| mean | 8.61/ 9.39 | 8.14/ 9.63 | 0.73 | 0.24 | 0.08 | 0.04 | 1.08 |
| S.D. | 3.34/ 2.63 | 4.29/ 2.83 | 1.57 | 0.85 | 0.55 | 0.19 | 2.62 |

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Question 7 (recognising a derivative)
Of the three choices, 2 is correct, 3 & 4 are incorrect, with 4 having graph & derivative of similar shape.
Codes are c:correct, i:incorrect, s:'same shape' (incorrect), ?:researcher unsure, -:no explanation.

| response: | 2 | 3 | 4 | nr |
|---|---|---|---|---|
| | 4 | 3 | 17 | 27 |

correct responses with reason................0 out of 51 (0%)

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Question 8 (specifying a non-differentiable function)

Codes are :f:formula given, g:graph drawn, e:written explanation only (all considered satisfactory)
if:incorrect formula, ig:incorrect graph, ie:incorrect explanation, -:no response (all unsatisfactory)

Satisfactory responses......................0 out of 51 (0%)

Table 9.10

First Year University Students on arrival

No. of students:44

| question | 1pre/post | 5pre/post | 6a | 6b | 6c | 6d | total | Q7 | Q8 | comments on Q6 (if any) |
|---|---|---|---|---|---|---|---|---|---|---|
| max. | 12/12 | 12/12 | 5 | 5 | 5 | 5 | 20 | | | |
| U01 | x/12 | x/12 | 5 | 5 | 5 | 5 | 20 | 2c | - | |
| U02 | x/10 | x/12 | 5 | 5 | 5 | 5 | 20 | 2c | - | |
| U03 | x/11 | x/12 | 5 | 5 | 5 | 1 | 16 | 2c | if | |
| U04 | x/11 | x/12 | 5 | 5 | 5 | 4 | 19 | 3i | - | |
| U05 | x/12 | x/12 | 5 | 5 | 5 | 5 | 20 | 2c | if | |
| U06 | x/8 | x/12 | 5 | 5 | 5 | 5 | 20 | 2i? | - | |
| U07 | x/8 | x/12 | 5 | 5 | 5 | 0 | 15 | 3i | - | |
| U08 | x/10 | x/11 | 5 | 5 | 5 | 4 | 19 | 2c | f | |
| U09 | x/10 | x/10 | 5 | 5 | 5 | 5 | 20 | 2c | f | including four formulae |
| U10 | x/11 | x/10 | 5 | 5 | 5 | 5 | 20 | 2c | f | |
| U11 | x/12 | x/12 | 5 | 0 | 0 | 0 | 5 | 4- | - | |
| U12 | x/10 | x/12 | 5 | 5 | 5 | 4 | 19 | 2c | - | |
| U13 | x/11 | x/12 | 5 | 5 | 5 | 5 | 20 | 2c | f | |
| U14 | x/9 | x/10 | 5 | 5 | 0 | 0 | 10 | 4- | if | |
| U15 | x/12 | x/12 | 5 | 5 | 0 | 0 | 10 | 2- | - | |
| U16 | x/11 | x/12 | 3 | 5 | 5 | 5 | 18 | 2c | if | |
| U17 | x/11 | x/12 | 5 | 5 | 5 | 4 | 19 | 2c | if | |
| U18 | x/12 | x/12 | 5 | 5 | 5 | 5 | 20 | 2c | - | |
| U19 | x/11 | x/12 | 5 | 5 | 5 | 5 | 20 | - | - | |
| U20 | x/11 | x/12 | 5 | 3 | 5 | 5 | 18 | 2c | f | |
| U21 | x/12 | x/12 | 5 | 5 | 5 | 5 | 20 | 2c | fg | |
| U22 | x/12 | x/12 | 5 | 5 | 5 | 5 | 20 | 2c | - | |
| U23 | x/12 | x/12 | 5 | 5 | 5 | 5 | 20 | 2i | - | including one formula |
| U24 | x/11 | x/10 | 5 | 5 | 5 | 5 | 20 | 2- | - | |
| U25 | x/12 | x/12 | 5 | 5 | 5 | 5 | 20 | 2? | - | |
| U26 | x/11 | x/10 | 5 | 5 | 5 | 1 | 16 | 2c | - | |
| U27 | x/11 | x/12 | 5 | 5 | 5 | 4 | 19 | 2c | - | including one formula |
| U28 | x/7 | x/12 | 5 | 5 | 5 | 5 | 20 | - | - | |
| U29 | x/11 | x/12 | 5 | 5 | 5 | 5 | 20 | 2c | f | |
| U30 | x/11 | x/12 | 2 | 1 | 2 | 1 | 6 | ?? | - | |
| U31 | x/9 | x/10 | 5 | 5 | 3 | 5 | 18 | 2c | - | |
| U32 | x/12 | x/12 | 5 | 4 | 3 | 0 | 12 | 2c | - | one graph shape |
| U33 | x/12 | x/12 | 5 | 5 | 5 | 5 | 20 | 2c | - | |
| U34 | x/11 | x/12 | 5 | 5 | 5 | 4 | 19 | 4i | - | |
| U35 | x/11 | x/12 | 5 | 5 | 5 | 5 | 20 | 2c | - | |
| U36 | x/11 | x/12 | 5 | 0 | 3 | 1 | 9 | 2c | - | two graph shapes |
| U37 | x/11 | x/12 | 5 | 5 | 5 | 5 | 20 | 2c | g | |
| U38 | x/11 | x/12 | 5 | 5 | 5 | 1 | 16 | 2c | if | one graph shape |
| U39 | x/10 | x/12 | 5 | 5 | 5 | 5 | 20 | 2c | g | |
| U40 | x/12 | x/12 | 5 | 5 | 5 | 5 | 20 | 2c | if | |
| U41 | x/11 | x/12 | 5 | 5 | 5 | 5 | 20 | 2c | if | |
| U42 | x/12 | x/12 | 5 | 5 | 5 | 5 | 20 | 2c | if | |
| U43 | x/11 | x/12 | 5 | 2 | 5 | 5 | 17 | 2c | if | |
| U44 | x/11 | x/12 | 5 | 5 | 5 | 5 | 20 | 2i | - | |

Total statistics overleaf

First Year University Students on arrival: fr... statistics

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| mean | -/10.86 | -/11.70 | 4.89 | 4.55 | 4.45 | 3.84 | 17.73 |
| S.D. | -/ 1.16 | -/ 0.69 | 0.53 | 1.27 | 1.37 | 1.86 | 3.93 |

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Question 7 (recognising a derivative)
Of the three choices, 2 is correct, 3 & 4 are incorrect, with 4 having graph & derivative of similar shape.
Codes are c:correct, i:incorrect, s:'same shape' (incorrect), ?:researcher unsure, -:no explanation.

| response: | 2 | 3 | 4 | nr |
|---|---|---|---|---|
| | 36 | 2 | 3 | 3 |

correct responses with reason................30 out of 44 (68%)

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Question 8 (specifying a non-differentiable function)

Codes are :f:formula given, g:graph drawn, e:written explanation only (all considered satisfactory)
if:incorrect formula, ig:incorrect graph, ie:incorrect explanation, -:no response (all unsatisfactory)

Satisfactory responses.......................9 out of 44 (20%)

Table 9.11

The major task of the chapter is to analyse these performances and to look closer at specific points which arise. For an analysis of the errors that occur, the whole population can be used to show the greatest variety of response, but to make comparisons to gain insight into the possible effects of the computer, a more careful matching is required.

**Selection of matched pairs**

Individuals in the experimental groups will be matched with students in the control groups who have the same (or better) scores on selected questions on the pre-test. The first question on the pre-test is a test of calculating numerical gradients. The mean scores attained by the various groups are given in table 9.12.

Pre test Q.1: mean scores (maximum 12)

| Experimental Groups | pre-test |
|---|---|
| KE | 10.5 |
| BE1 | 9.5 |
| BE2 | 10.6 |
| CE | 8.6 |

| Control Groups | pre-test |
|---|---|
| KC | 11.2 |
| BC1 | 8.7 |
| BC2 | 11.3 |
| BC3 | 9.3 |
| BC4 | 10.6 |

Table 9.12

It will be seen that the Cricklade group scores lower than any of the others and its performance will be considered separately.

The control groups BC4, BC2, KC, scored higher than or equal to BE2, KE, BE1 respectively, with BC3 and BC1 marginally lower. To match the pairs using this information in such a way that the control student in each pair performed as well or better on the pre-test, the students were placed in two lists: experimental in order BE2, KE, BE1 and control in order BC4, BC2, KC, BC3, BC1. Those in the experimental groups were considered in succession and matched with the first student in the control groups who obtained an equal or (failing that) greater score on the pre-test. Using this method, the control student in each pair would not only have at least as good a score, (s)he was also more likely to have come from a group whose average score was better.

For those who had not done calculus before, this was the only statistic considered. For those with previous calculus, their performance on the pre-test question on formal differentiation of polynomials and powers was also taken into account. Thus in each pair of students with previous calculus experience, the control student obtained equal or greater marks than the experimental student on both the numerical gradient question and the formal differentiation. The matched pairs chosen, and their marks on the pre-test are as in table 9.13 (with numerical gradient mark preceding that for formal differentiation):

<u>Students without previous calculus experience:</u>

| | | | |
|---|---|---|---|
| BE206 | (11) | KC07 | (11) |
| KE01 | (11) | KC04 | (11) |
| KE02 | (10) | BC105 | (10) |
| KE03 | (10) | KC09 | (11) |
| KE05 | (9) | BC113 | (9) |
| KE06 | (12) | KC01 | (12) |
| KE08 | (12) | KC02 | (12) |
| KE09 | (10) | BC112 | (11) |
| KE11 | (11) | BC115 | (11) |
| KE12 | (12) | KC06 | (12) |
| KE13 | (12) | KC08 | (12) |
| KE14 | (6) | BC107 | (8) |

<u>Students with previous calculus:</u>

| | | | |
|---|---|---|---|
| BE101 | ( 9/12) | BC204 | (10/12) |
| BE102 | (11/12) | BC206 | (11/12) |
| BE103 | (11/12) | BC217 | (11/12) |
| BE104 | ( 0/12) | BC311 | ( 7/12) |
| BE105 | (12/7 ) | BC315 | (12/8 ) |
| BE106 | (11/12) | BC404 | (11/12) |
| BE107 | (12/7 ) | BC110 | (12/9 ) |
| BE108 | (10/9 ) | BC412 | (10/10) |
| BE109 | ( 6/12) | BC111 | ( 7/12) |
| BE110 | (11/12) | BC406 | (11/12) |
| BE112 | (10/12) | BC405 | (10/12) |
| BE113 | (11/12) | BC407 | (11/12) |
| KE07 | (12/8 ) | BC202 | (12/10) |
| KE15 | (10/8 ) | BC216 | (10/8 ) |
| KE16 | (12/9 ) | BC218 | (12/11) |
| BE201 | (12/10) | BC207 | (12/12) |
| BE202 | (10/6 ) | BC410 | (11/6 ) |
| BE203 | ( 9/12) | BC408 | (11/12) |
| BE204 | (11/6 ) | BC214 | (11/7 ) |
| BE205 | (12/8 ) | BC213 | (12/12) |
| BE207 | ( 6/4 ) | BC103 | ( 6/4 ) |
| BE208 | (12/10) | BC402 | (12/12) |
| BE209 | (12/12) | BC303 | (12/12) |
| BE211 | ( 6/0 ) | BC102 | ( 8/0 ) |
| BE214 | (11/4 ) | BC215 | (11/4 ) |
| BE215 | ( 9/10) | BC212 | (11/10) |
| BE216 | (12/8 ) | BC208 | (12/12) |

<u>Table 9.13</u>

As a check on this allocation of pairs, the second question on the pre-test was used to see if the performance of the pairs differed significantly at the 5% level.

Question 2 part 1 required the gradient of the chord through (1,1), $(k,k^2)$. One mark was given for the response $(k^2-1)/(k-1)$ (even if there were subsequent errors) and 2 marks for reducing it to the form k+1. The relative performance on the matched pairs without calculus (experimental minus control) were as follows:

0 0 0 0 +1 +1 0 0 −1 +1 +1 0.

The null hypothesis is that the number of positive and negative signs is the same. With 1 out of 5 signs negative the probability of attaining this result, or more extreme, on a two tail-test sign test is $p=0.38$. The null hypothesis is therefore rejected. There is a small but statistically insignificant trend for the experimental students to perform better on this question in the pre-test.

The same analysis for the students with previous calculus experience gives:

0 +1 −1 +1 0 −1 +1 −1 −1 −1 −2 +1 +1 0
0 0 +1 0 +1 0 0 −1 0 0 0 −2 −1.

Here 9 of the 16 signs are negative and the probability of attaining this, or a more extreme case, on a two-tail test is $p \simeq 0.8$.

The remaining two parts of question 2 may also be compared for significant differences. Part (ii) requests the gradient of the tangent to $y=x^2$ at (1,1). Giving 1 mark for obtaining the correct answer 2 produces the following differences on the pre-test:

<u>no previous calculus</u>: +1 0 0 0 0 −1 −1 +1 0 +1 0 0
(2 negative signs out of 5, giving p=1.)

<u>previous calculus</u>: −1 0 0 0 0 0 0 0 0 0 +1
0 +1 −1 0 0 +1 −1 −1 0 0 0 +1 +1 0 0 0
(4 negative signs out of 9, p=1).

Finally, classifying the responses to the explanation of finding the gradient of the tangent from first principles according to whether the responses are "static", "pre-dynamic" or "dynamic", the differences between the control and experimental groups on the pretest (using a marking scheme to be discussed in detail in chapter 10) allows a Wilcoxon matched-pairs signed-ranks test to be used as in Siegal [1956]. The null hypothesis $H_0$ is that $\Sigma P$ (the sum of the rankings of the positive scores) equals $\Sigma N$ (the sum of the rankings of the negative scores). The test statistic T is the smaller of $\Sigma N$ and $\Sigma P$. (See Siegal for details). The statistics found are:

<u>no previous calculus</u>: -1 -1 0 0 +1 0 0 0 0 0 0 -1

$H_o$:$\Sigma P=\Sigma N$, N=4, T=2.5: reject $H_o$ ($p \approx 0.6$).

<u>previous calculus</u>: +2 +1 -1 +2 -5 -6 +1 -1 -2 -3 +2 -1 +2 -2 +4 0 0 -6 -1 +2 +1 -6 +2 +1 -1 +4 +1

$H_o$: $\Sigma P=\Sigma N$, N=26, T=155.5: reject $H_o$ ($p \approx 0.85$).

All the comparisons made between the control and experimental groups are therefore statistically insignificant. However, there may be problems caused by the small size of the groups with no previous calculus experience; these will require a large change in performance to yield a statistically significant improvement.

**Task (A): Calculating numerical gradients from a picture**

The work of Orton [1980a] indicated that students might have serious weaknesses in the basic ideas of rate of change which is fundamental to the notion of derivative. In particular the calculation of "... the average rate of change of y with respect to x" between points on a curved graph showed that 35% of a group of 110 mixed ability students gave a wrong answer when the x and y increments were each one! [Orton, 1980b page 207.] In mitigation, the graph did not have equal scales and the students may not have understood the term "average rate of change", even so, the possibility of such a fundamental misconception would pose grave difficulties in the learning of the calculus.

The first question on the pre- and post-tests was of a similar kind, but had equal scales and gave an explanation of the "average rate" in terms of gradient. (Figure 9.14.)

1.Find the average rate of change between the following points on the graph:
*(Note: the "average rate of change" from P to Q means the gradient of PQ)*

(i) from C to D .....
(ii) from D to E .....
(iii) from A to B .....
(iv) from B to C .....
(v) from C to E .....
(vi) from D to C .....

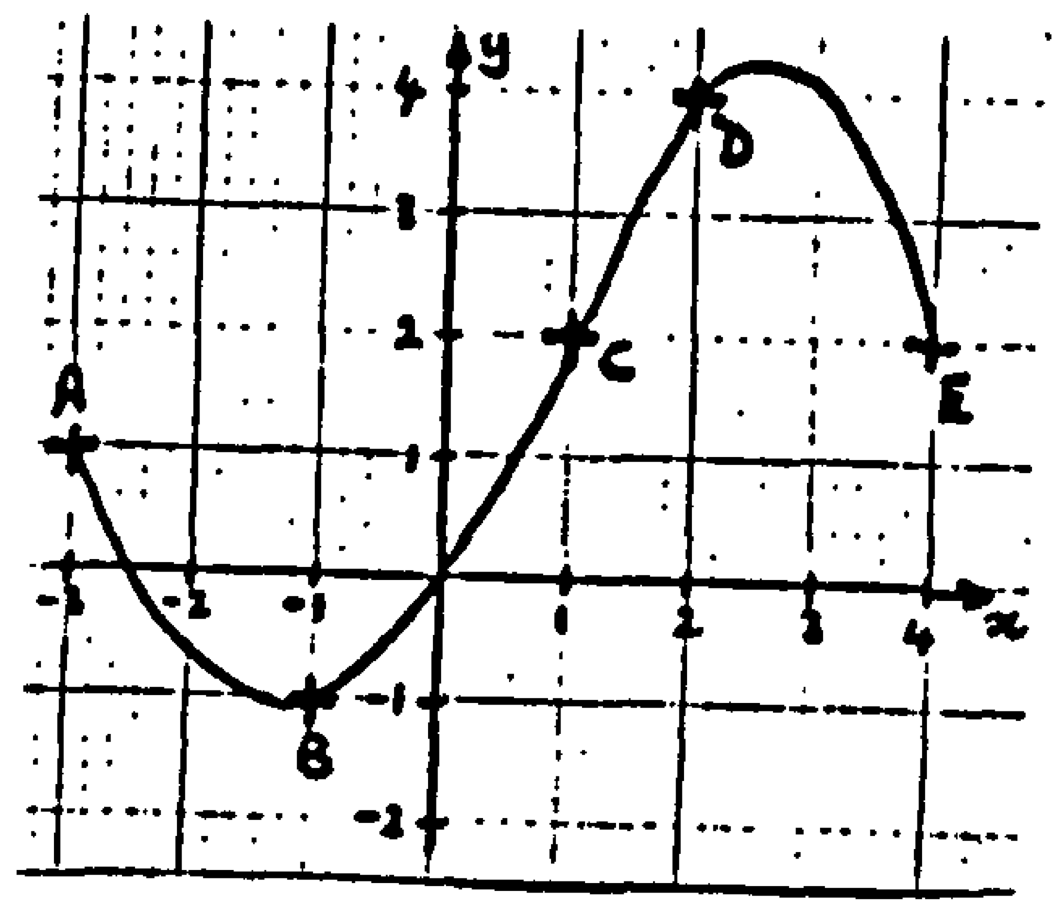



Figure 9.14

These results were more encouraging. Giving two marks per question, with one mark for a correct numerical value with an incorrect sign, the mean scores of the groups were as given earlier in table 9.12.

Analysis of errors in task (A)

The marks obtained by students in different classes are at first sight not very informative. However, the percentage of errors made on each question by the all the students taking pre- and post-tests does reveal interesting information (Table 9.15).

Q.1: Percentage of errors on each question

| | (i) | (ii) | (iii) | (iv) | (v) | (vi) |
|---|---|---|---|---|---|---|
| Experimental (N=42) | | | | | | |
| Pre | 7 | 17 | 19 | 19 | 12 | 45 |
| Post | 2 | 5 | 5 | 5 | 5 | 52 |
| Control (N=67) | | | | | | |
| Pre | 7 | 22 | 24 | 12 | 15 | 51 |
| Post | 3 | 16 | 13 | 4 | 1 | 52 |
| Cricklade (N=51) | | | | | | |
| Pre | 18 | 29 | 47 | 27 | 27 | 59 |
| Post | 6 | 33 | 37 | 24 | 12 | 57 |
| University (N=44) | | | | | | |
| Pre | - | - | - | - | - | - |
| Post | 0 | 18 | 7 | 0 | 2 | 57 |

Table 9.15

Task (i) (with gradient 1) and (v) (with gradient zero) were in general answered well, with task (iv) (a positive gradient crossing the axes) almost as well done in the post-test. But tasks (ii) and (iii) with a negative y-step have a higher percentage of errors and task (vi), with negative x-step and negative y-step, has half the students giving a wrong answer each time.

An analysis of errors caused by an incorrect sign only is given in table 9.16.

Q.1: Percentage of errors through a mistake in sign:

| | (i) | (ii) | (iii) | (iv) | (v) | (vi) |
|---|---|---|---|---|---|---|
| Experimental (N=42) | | | | | | |
| Pre | 5 | 10 | 10 | 2 | 0 | 36 |
| Post | 0 | 2 | 2 | 0 | 0 | 48 |
| Control (N=61) | | | | | | |
| Pre | 0 | 16 | 18 | 0 | 0 | 46 |
| Post | 0 | 13 | 9 | 0 | 0 | 49 |
| Cricklade (N=51) | | | | | | |
| Pre | 2 | 14 | 16 | 6 | 0 | 39 |
| Post | 0 | 16 | 6 | 2 | 0 | 53 |
| University (N=44) | | | | | | |
| Pre | - | - | - | - | - | - |
| Post | 0 | 0 | 0 | 0 | 0 | 55 |

Table 9.16

The major error in the last part is therefore in deciding the sign of the gradient.

Table 9.17 shows the percentage of students whose response changes on each question, showing improvements and deteriorations from pre-test to post-test.

Q.1: % of students changing response from pre- to post-test

| | (i) | (ii) | (iii) | (iv) | (v) | (vi) |
|---|---|---|---|---|---|---|
| Experimental (N=42) | | | | | | |
| Better | 5 | 17 | 17 | 17 | 10 | 19 |
| Worse | 2 | 5 | 2 | 5 | 2 | 24 |
| Control (N=67) | | | | | | |
| Better | 7 | 15 | 18 | 12 | 13 | 18 |
| Worse | 3 | 7 | 10 | 4 | 0 | 16 |
| Cricklade (N=51) | | | | | | |
| Better | 14 | 18 | 25 | 16 | 20 | 29 |
| Worse | 2 | 24 | 18 | 14 | 4 | 18 |

Table 9.17

On each question a minority of students give a worse response on the post-test. However, only two students out of 160 failed to get at least two questions right out of six on the post-test, so there is evidence that the vast majority have some understanding of the concept. The responses suggest an underlying pattern of random error, especially in the weaker students.

The last question has a high degree of instability. Overall, 41% of those doing both pre- and post-test change their response, with almost equal numbers moving in each direction. Only 26% give a correct response both times.

It is unlikely that students will have met negative x-increments before and in this case they are faced with a conflict. The line is sloping *up* (suggesting a positive result), but the actual y-direction from D to C is *down* (suggesting a negative result).

Perhaps their feeling for positive and negative directions is stronger in the vertical direction than the horizontal. The net result is an unresolved conflict that persists in the most able students entering university to read mathematics.

Even though the experimental students were explicitly shown the concept of negative gradient, the computer program refers to the gradient of the chord through points (a,f(a)), (b,f(b)) but does not letter the diagram. When b is to the left of a, the dominant factor in the picture is the gradient of the chord, not the order of a,b. Thus students could have a good appreciation of the notion of positive and negative gradient without ever coming face to face with the situation in (vi). This would account for the fact that the use of the computer seems to have little effect on performance in this task.

### Comparison of matched pairs on numerical gradients (task (A))

One may conjecture that the experimental students, exposed to the computer carrying out numerical calculations and given the opportunity to discuss the ideas, would perform better than the control students. As these performances are matched in pairs on the pre-test, the improvement may be tested by a one-tailed Wilcoxon test. The statistics comparing the experimental and control scores on the post-test are as follows:

<u>no previous calculus</u>: 0 0 0 −1 +1 +1 0 −1 +1 0 0 +1

($H_o$: $\Sigma P \leq \Sigma N$, N=6, T=7. Accept $H_o$, no significant improvement, $p \simeq 0.4$.)

<u>with previous calculus</u>: −5 +3 0 −1 0 0 0 0 +1 +1 0 0 +3 +1 0 +1 +3 +2 −1 +3 0 +1 +1 +4 0 −1 +1

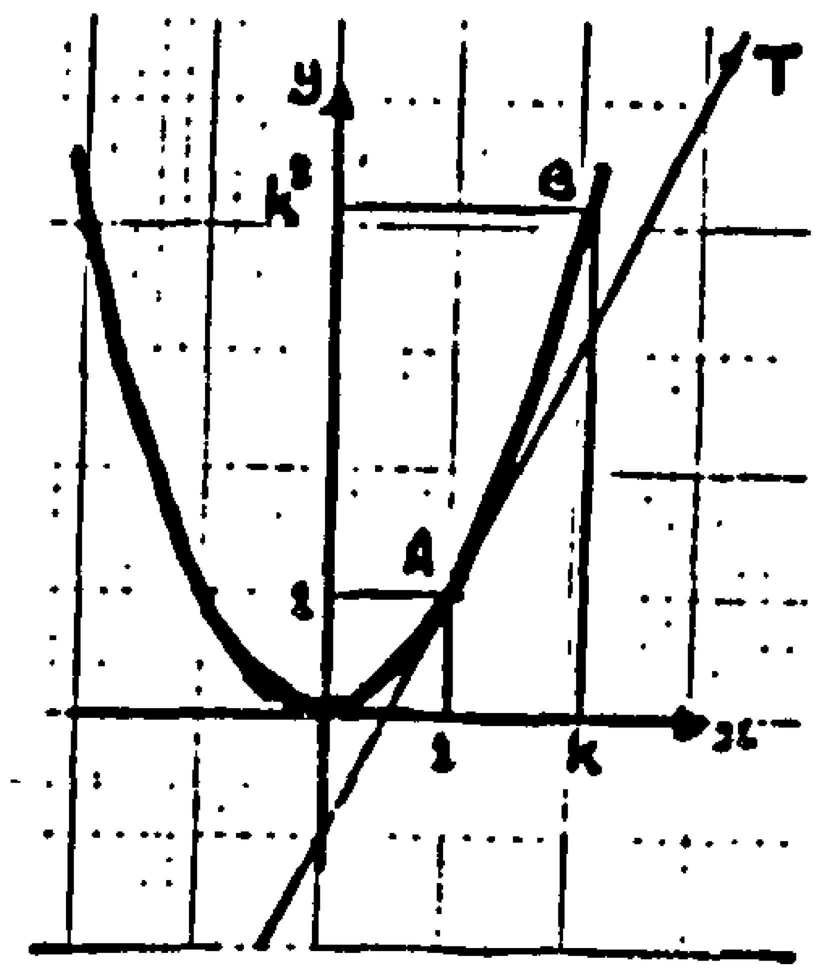
($H_o$: $\Sigma P \leq \Sigma N$, N=17, T=28. Reject $H_o$, a significant improvement, $p<0.025$.)

The smaller groups without previous calculus experience show a small, but statistically insignificant, improvement whilst those with previous calculus experience show an improvement which is statistically significant at the 2.5% level. However, one should not set too much store by this result. It occurs on a question which shows only a small improvement from pre- to post-test overall, subject to conflicting changes in performance on each test.

Task (B): differentiation from first principles

The second question on pre- and post-test (Figure 9.18) was intended to study whether the notion of a chord tending to a tangent was *a priori* a spontaneous method of solution, and whether the experimental groups in any way differed in their response from the controls.

2.




On the graph $y=x^2$, the point A is (1,1), the point B is $(k,k^2)$ and T is a point on the tangent to the graph at A.

(i) Write down the gradient of the straight line through A,B....

(ii) Write down the gradient of AT.....

Explain how you might find the gradient of AT from first principles.

Figure 9.18

The responses to this question reveal only *evoked* concept images in the sense that if the question were re-phrased, other responses may occur. The first part of the question tested the student's response to the form of the gradient, which could be written as

$(k^2-1)/(k-1)$

and possibly simplified to give

$k+1$.

In the latter form the student may let $k \to 1$ so that the gradient tended to 2. The last part of the question was intended to give information as to whether the student could give some explanation of this process.

The responses to the first two parts are recorded in table 9.19. As some of the groups are rather small and percentages may be inappropriate, the actual numbers in each category are given. The figures in brackets in the first two columns indicate the number of students who obtain the appropriate formula and also give the gradient of the tangent as 2. Some of these students may be expected to let $k \to 1$ and so derive the gradient of the tangent using a limiting process. The symbol "nr" denotes "no response".

| Response: | (i): $k+1$ | $\frac{k^2-1}{k-1}$ | other | nr | (ii): 2 | other | nr |
|---|---|---|---|---|---|---|---|
| Experimental (without previous calculus experience) (N=12) | | | | | | | |
| Pre | 1(1) | 6(5) | 5 | 0 | 7 | 4 | 1 |
| Post | 3(3) | 5(2) | 4 | 0 | 11 | 1 | 0 |
| Control (without previous calculus experience) (N=15) | | | | | | | |
| Pre | 0(0) | 7(2) | 7 | 1 | 8 | 4 | 3 |
| Post | 0(0) | 4(3) | 11 | 0 | 9 | 4 | 2 |
| Cricklade (without previous calculus experience) (N=37) | | | | | | | |
| Pre | 5(2) | 12(2) | 20 | 0 | 11 | 15 | 11 |
| Post | 0(0) | 14(4) | 19 | 4 | 19 | 8 | 10 |
| Experimental (with previous calculus experience) (N=30) | | | | | | | |
| Pre | 9(6) | 10(9) | 11 | 0 | 22 | 7 | 1 |
| Post | 10(8) | 4(3) | 15 | 1 | 26 | 3 | 1 |
| Control (with previous calculus experience) (N=52) | | | | | | | |
| Pre | 8(7) | 30(24) | 11 | 3 | 38 | 8 | 6 |
| Post | 9(6) | 24(18) | 18 | 1 | 38 | 9 | 5 |
| Cricklade (with previous calculus experience) (N=14) | | | | | | | |
| Pre | 0(0) | 7(2) | 7 | 0 | 4 | 9 | 1 |
| Post | 0(0) | 10(6) | 3 | 1 | 7 | 3 | 4 |
| University (with calculus experience) (N=44) | | | | | | | |
| Post | 25(23) | 16(13) | 3 | 0 | 40 | 4 | 0 |

Table 9.19

Although there are small increases from pre-test to post-test in students obtaining the gradient 2 for the tangent in part (ii), there is an overall *decrease* in students obtaining the appropriate formula for the chord in part (i). A number of these are through calculating numerical values for the gradient of the chord, misled no doubt into giving k the numerical value 4.1 or thereabouts. Clearly the grid on the picture was a distractor.

Counting up the figures in brackets, on the pre-test 16 students (10%) obtained the value k+1 for the chord and 2 for the tangent, and 44 (28%) obtained $(k^2-1)/(k-1)$ and 2. On the post-test 17 students (11%) obtained $k+1$ and 2, whilst 38 (24%) obtained $(k^2-1)/(k-1)$ and 2. Thus 60 students (38%) on the pre-test and 55 (34%) on the post-test were in a possible position to report that, as $k \to 1$, so the chord gradient tended to 2. At the university the proportion was considerably higher: 36 out of 44 (82%). But how many students allowed k to tend to 1 to find the gradient of the tangent? An analysis of the responses shows only one student on the pre-test and one other on the post-test. This will be considered in greater detail in Chapter 10.

### Task (C): Comparison of matched pairs on formal differentiation

One would not expect the computer programs to have any effect on the ability of students to carry out the formal differentiation algorithm. At this stage of development the students had only covered derivatives of polynomials and powers. The derivatives of the following were requested on both pre- and post-test:

(a) $x^4+3x^2$

(b) $\sqrt{x}$

(c) $1/x^2$.

These tested the knowledge of the derivative of $x^n$ and the formal handling of fractional and negative powers. Each question was marked out of four; if the correct formula was obtained followed by a single error, three marks were given. Any other response (which never obtained a fully correct formula) was given a maximum of two marks, with one mark deducted for each further error.

The relative performance of the matched pairs on the post-test was as follows:

<u>without previous calculus</u>: -2 +2 -5 +4 +4 +6 -5 -6 0 +6 +6 +2

$N=11$, $T=25$, $H_0$: $\Sigma P=\Sigma N$. Accept null hypothesis, no significant difference ($p \doteq 0.48$).

<u>with previous calculus</u>: 0 +12 0 +1 0 0 -2 -2 0 0 0 0 0 0 0 0 -4 +2 0 +2 -4 0 0 +4 +2 0 -4

$N=11$, $T=29$, $H_0$: $\Sigma P=\Sigma N$. Accept null hypothesis, no significant difference ($p \doteq 0.72$).

Although there is a marginal improvement by the experimental students, this is not statistically significant in either group.

**Task (D): Sketching the derivative of a function with given graph**

A difference that one would expect is in the experimental students' visualization of the gradient of the graph. For example, if the students are given the graph of a function which looks fairly smooth and are asked to draw the derivative of the function, those with only the formal algorithm for differentiation may go through the following process:

(a) attempt to spot the formula of the graph

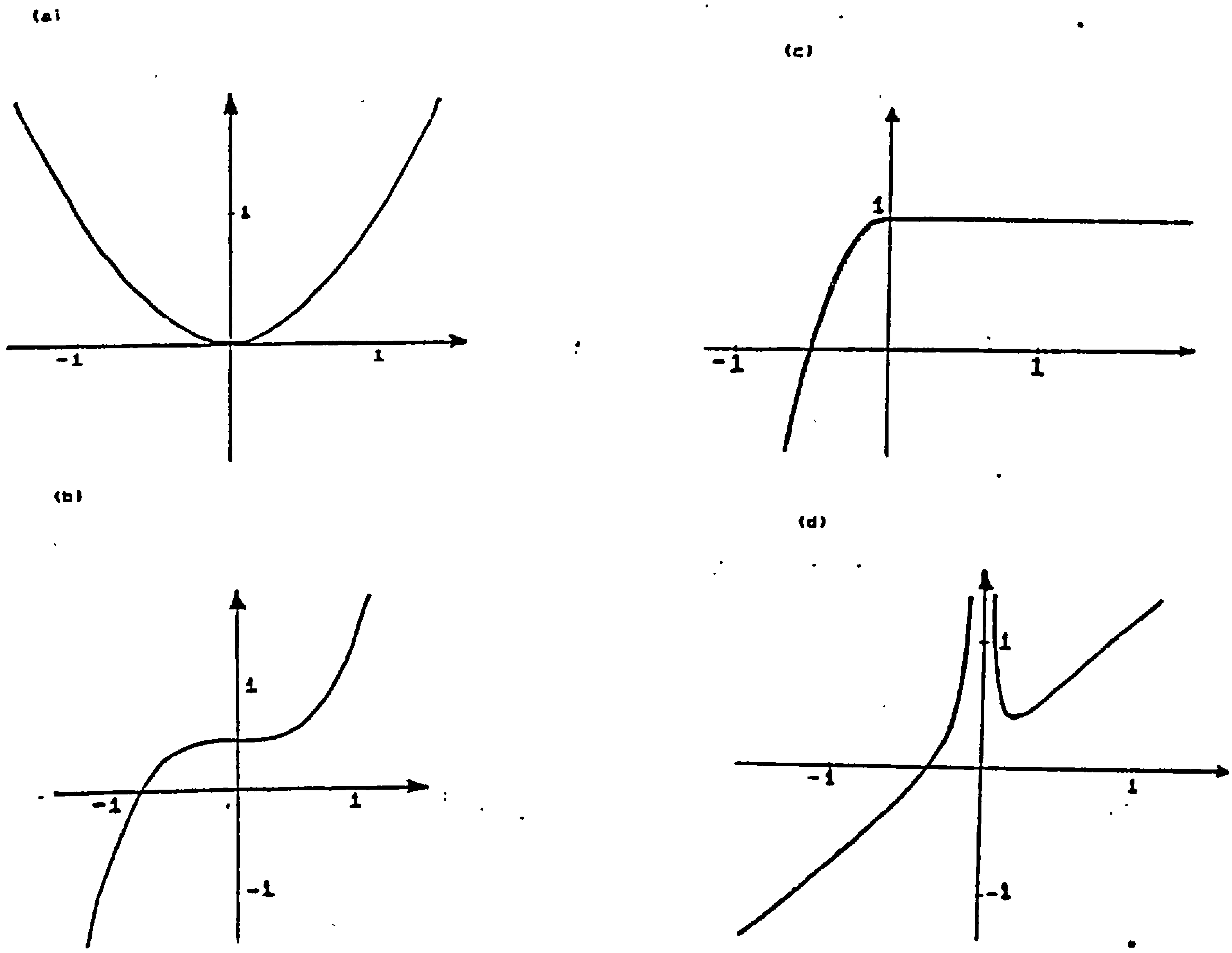
(b) differentiate the formula

(c) draw the graph of the result.

However, those who had used the computer programs may have the dynamic mental imagery of a chord clicking along the graph and be able to scan the graph, visualize the gradient function, then draw a sketch of their mental image.

The next part of the post-test gave a sequence of four graphs of increasing difficulty for the students to sketch the derivative (figure 9.20)

6. Sketch the *derivatives* of the following graphs:




<u>figure 9.20</u>

The first looks like $y=x^2$ (it is actually $y=abs(x^{1.8})$. Thus a student without the global mental image of the gradient function could say "it looks like the graph of $y=x^2$, the derivative is $y=2x$, so draw the latter graph". The second graph is $y=x^3+1$, which is harder to guess. The third is not a graph with an easily recognisable formula, so the algorithmic approach is not readily available. But the dynamic idea of the gradient function makes it relatively easy to visualize the derivative. The fourth graph is less easy from either point of view.

One would hope that the experimental students would perform better than the control student on all four tasks, with their performance even better on the later ones where the algorithmic approach is not readily available. Five marks were awarded for each graph, based on the essential factors of the gradient graph. For example on the first graph there are three main features: negative and increasing gradient first (2 marks), zero gradient at the origin (1 mark), then increasing positive gradient (2 marks).

The marks awarded in each of the groups show significant trends that hardly require statistics to see the improvement of experimental over control. Table 9.21 shows the average mark in each group on the four questions (maximum 5 marks each) and total (maximum 20). The experimental and control groups are in descending order of rank according to marks attained on table 9.1.

Sketching a derivative for a given graph

| Graph | (a) | (b) | (c) | (d) | total |
|---|---|---|---|---|---|
| Maximum | 5 | 5 | 5 | 5 | 20 |
| BE2 | 4.62 | 4.25 | 4.44 | 3.94 | 17.25 |
| KE | 4.93 | 4.50 | 4.36 | 4.07 | 17.86 |
| BE1 | 5.00 | 3.92 | 3.50 | 2.00 | 15.83 |
| BC4 | 4.55 | 3.18 | 2.82 | 2.18 | 12.73 |
| BC2 | 4.11 | 3.78 | 1.56 | 0.94 | 10.39 |
| KC | 3.67 | 4.00 | 0.78 | 0.00 | 8.44 |
| BC3 | 2.36 | 1.71 | 0.14 | 0.07 | 4.29 |
| BC1 | 1.93 | 0.87 | 0.00 | 0.20 | 3.00 |
| CE | 0.73 | 0.24 | 0.08 | 0.04 | 1.08 |
| U | 4.89 | 4.55 | 4.45 | 3.84 | 17.73 |

Table 9.21

Visibly the experimental groups perform better than the control groups, at a level comparable with that of the best students arriving to study mathematics as a major subject at university. The Cricklade students are hardly able to make any contribution at all.

Almost any statistical test can be employed to show that the experimental students perform better than the control students. For example, the experimental group scoring lowest on this test scores higher than the highest control group (though an analysis of covariance shows this to be not statistically significant).

If the experimental and control students are each divided into three groups according to their performance on the pre-test numerical gradients (question 1): high (scoring 11 or 12), medium (scoring 7-10) and low (scoring 0-6), then an analysis of covariance shows the experimental groups scoring higher than the corresponding control groups at a highly significant level ($p<0.001$ in each case). An analysis of covariance also shows that the *low experimental* group outscores the *high control* group at the 5% level.

**Comparison of matched pairs (Task (D))**

To obtain a more reliable comparison matched pairs may be used. The scores on each question will be compared using the null-hypothesis for a one-tail Wilcoxon test ($H_o: \Sigma P \leq \Sigma N$). The statistics are as follows (experimental minus control):

Graph (a)

Without previous calculus:

0 -1 +5 -1 +5 0 +5 0 0 0 +5

N=6, T=3. Accept null hypothesis. Not (quite) significant ($p \simeq 0.06$).

(But note small sample.)

With previous calculus:

0 +5 0 +4 0 0 +5 0 0 0 0 0 0 0 +3 0 +5 0 0 0 0 0 0 +5 -1 +5 0

N=8, T=1. Reject $H_o$: significant improvement ($p<0.01$).

Graph (b)

Without previous calculus:

+5 +5 +5 0 +5 +5 +5 +3 +5 +5 +5 +3

N=11, T=0. Reject $H_o$: significant improvement ($p<0.001$).

With previous calculus:

-2 +5 0 -3 +5 0 +5 +5 +1 0 0 0 +2 +1 0 0 +3 0 0 +1 +3 0 +5 +5 -2 +2 -2

N=17, T=25.5. Reject $H_o$: significant improvement ($p<0.001$).

Graph (c)

Without previous calculus:

+2 +5 +5 0 +4 +5 +5 0 +1 +5 +5 +3

N=10, T=0. Reject $H_o$: significant improvement ($p<0.001$).

With previous calculus:

-3 +5 0 +2 +5 -3 +5 +5 0 0 0 0 +5 0 +5 +3 +5 +1 +2 +3 +4 +5 0 +5 +5 +1 +5

N=20, T=13. Reject $H_o$: significant improvement ($p<0.001$).

Graph (d)

Without previous calculus:

+2 +5 +5 0 +4 +5 +5 0 +1 +5 +5 +3

N=10, T=0. Reject $H_0$. Significant improvement ($p<0.001$).

<u>With previous calculus:</u>

-1 +5 0 0 +5 -4 +4 -1 +1 0 +3 0 +5 +2 +5 +2 +5 -1 +1 +5 +4 -1 +3 +5 +4 +5 +1

N=23, T=29.5 Reject $H_0$: significant improvement ($p<0.001$).

Thus in every case, except graph (a) with the smaller groups lacking previous calculus experience, the experimental students score significantly higher than the control. The difference on graphs (c) and (d), where the route using formal calculus formulae is considerably more difficult, is significant at the 0.1% level: it could only occur by chance in less than one trial in a thousand.

Task (E): Recognising a derivative

The next question on the post-test drew the graph of a derivative (a) and gave the choice of three graphs (b), (c) and (d) to select the *original* graph (figure 9.22).

7. Graph 1 is the *derivative* $y=f'(x)$ of a function $y=f(x)$ defined for $0 \le x \le 8$.

Which of the graphs 2,3,4 could be the *original* graph $y=f(x)$?

Give the reason(s) for your choice:

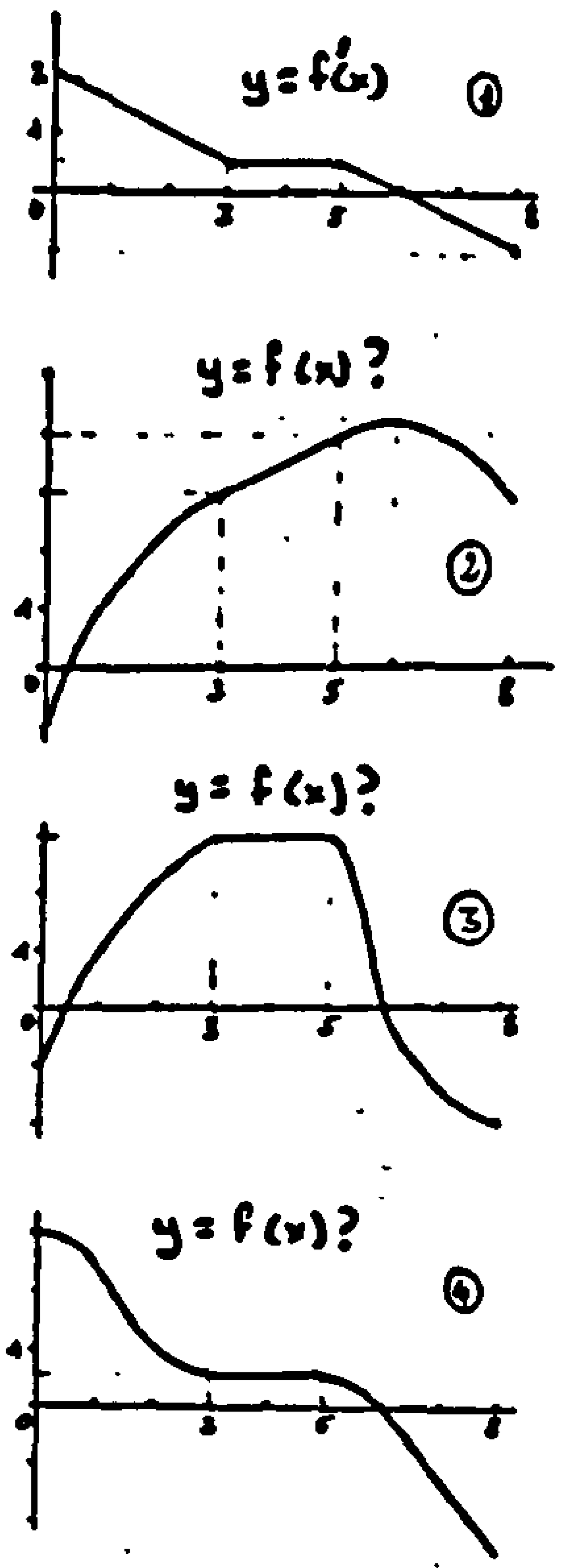



Figure 9.22

The correct choice is (b)*, with (d) being a similar outline shape to (a), providing a possible distractor. The performance of the various groups again followed a similar pattern (table 9.23). Here "(b)+" denotes the response (b) together with a correct reason, (b),(c),(d) denote the corresponding responses and nr denotes "no response".

* In the text, graphs (a),(b),(c),(d) respectively refer to 1,2,3,4 in figure 9.22.

Recognising a derivative

| Response: | (b)+ | (b) | (c) | (d) | nr |
|---|---|---|---|---|---|
| BE2(N=16) | 12 | 13 | 1 | 2 | 0 |
| KE (N=14) | 10 | 11 | 2 | 1 | 0 |
| BE1(N=12) | 6 | 7 | 3 | 2 | 0 |
| BC4(N=11) | 4 | 7 | 2 | 2 | 0 |
| BC2(N=18) | 2 | 5 | 3 | 7 | 3 |
| KC (N=9) | 3 | 3 | 2 | 3 | 1 |
| BC3(N=14) | 3 | 5 | 3 | 3 | 3 |
| BC1(N=15) | 0 | 0 | 5 | 9 | 1 |
| CE (N=51) | 0 | 4 | 3 | 17 | 27 |
| U (N=44) | 30 | 36 | 2 | 3 | 3 |

Table 9.23

Once more the better response of the experimental groups over the control groups is visible, with 67% of the experimental students in category (b)+, as against only 8% of the control students. The Cricklade group performs visibly worse: none of the 51 students there gave a correct response with a satisfactory reason and more than half failed to respond. Of those who did, 17 out of 24 (70%) chose the distractor with the same shape as the original graph.

On the other hand the experimental students are again performing at a level comparable with that of students taking mathematics as their major subject at university (with 68% in category (b)+).

Comparison of Matched Pairs (Task (E))

Awarding 1 mark for the correct response (b) and 1 mark for a correct reason, giving a maximum of 2, comparison of experimental and control students yields the following Wilcoxon statistics for a one-tail test ($H_o: \Sigma P \leq \Sigma N$):

<u>Without Previous Calculus</u>: 0 0 +2 0 +2 0 +2 +1 +2 0 +2 +2

N=7, T=0. Reject $H_o$, significant improvement ($p<0.01$).

<u>With Previous Calculus</u>:0 +2 0 -2 +2 -2 +1 0 0 0 +1
+2 +2 +2 +1 0 +2 0 +1 0 0 +1 +2 +2 +2 +2 +2

N=18,T=24. Reject $H_o$, significant improvement ($p<0.01$).

**Task (F): specifying a non-differentiable function**

It is natural that one would expect the experimental students to do better than the control students in this case, for the control students would be unlikely to have discussed the idea at all. Students were asked:

> Give an example of a function which is defined at x=1 but is not differentiable at x=1.

The number of correct responses (allowing either a formula, a drawing of an appropriate graph, or a satisfactory explanation) were as follows (table 9.24):

Specifying a non-differentiable function

| | correct | incorrect | nr |
|---|---|---|---|
| BE2 (N=16) | 2 | 12 | 2 |
| KE (N=14) | 7 | 5 | 2 |
| BE1 (N=12) | 6 | 4 | 2 |
| BC4 (N=11) | 0 | 3 | 8 |
| BC2 (N=18) | 0 | 8 | 10 |
| KC (N=9) | 1 | 2 | 6 |
| BC3 (N=14) | 0 | 4 | 10 |
| BC1 (N=15) | 0 | 5 | 10 |
| CE (N=51) | 0 | 9 | 42 |
| U (N=44) | 9 | 10 | 25 |

Table 9.24

There was a much lower level of correct response on this question than on the previous ones. Only one control student managed to concoct an appropriate formula, $y=\sqrt{(x-1)}$ (which has a vertical tangent on one side only). In each of the experimental groups KE and BE1, half the students gave a correct answer, but only two out of sixteen in BE2. An analysis of the teacher's diary reported in the previous chapter explains the low success rate in BE2: the notion was only discussed in passing in one session. The result was that, although 14 out of 16 responded to the question, only 2 of them were correct. The other twelve had a sense of what was required but were unable to articulate it correctly.

By contrast, two thirds of the control groups gave no response and at Cricklade the figure rose to 82%. Even amongst the students who had arrived at university to read mathematics, more than half (57%) failed to respond.

Comparison of matched pairs (Task (F))

Giving 3 marks to a correct formula, 2 marks to a satisfactory graph and 1 mark for a satisfactory general explanation, comparison of experimental and control students yields the following Wilcoxon statistics for a one-tail test ($H_o: \Sigma P \leq \Sigma N$):

<u>Without Previous Calculus</u>: 0 0 +3 0 0 +2 +2 0 +2 +1 +2 +2

N=7, T=0. Reject $H_o$, significant improvement ($p<0.01$).

<u>With Previous Calculus</u>: +1 +2 +2 0 +1 0 +2 0 0 0 +1 0 0 0 0
0 0 0 0 +2 0 0 0 0 0 0 0

N=6,T=0. Reject $H_o$, significant improvement ($p<0.05$).

This "significant" result should be taken with a pinch of salt. What it shows is that if students are shown examples of non-differentiable functions at the outset, some of them are able to remember what they have seen. It shows also that those who have not discussed non-differentiability have little chance of handling it satisfactorily when asked to do so out of context. If the idea is introduced briefly using the computer (as in group BE2), it may be that the students have some idea of the concept (in terms of the number willing to "have a go") but they may not able to cope with it at any constructive level, for example through inventing (or even remembering!) simple non-differentiable examples.

## Control students who perform well

At this stage it is useful to look at those individuals who did not use the computer, yet performed well on the tasks involving visualizing the derivative as a gradient function. As a cut-off point, we consider those students scoring 16 or more marks on the question sketching the derivative for a graph. This range of marks was achieved by 32 out of the 42 experimental students. It was only achieved by 7 out of 67 control students:

BC205, BC216, BC217, BC402, BC406, BC407, BC408.

These come from the two groups which score highly on numerical gradients and formal differentiation; BC4 is the only group consisting entirely of students doing both mathematics and further mathematics at A-level. All these students had studied the calculus before at O-level. Clearly some able students are capable of forming the mental concept of the global gradient of a graph after some experience of standard calculus. This is in accord with the high performance on this question of the able students reading mathematics at university. However, only two of these seven students were able to recognise a derivative and to give a clear reason for their choice. Thus only two out of 72

(3%) control students performed well on both tests, as compared with 26 out of 42 experimental students (62%).

## Experimental students who perform badly

Here we consider those students scoring less than half marks in sketching derivatives. At Kenilworth and Barton Peveril, these consist of five individuals:

KE04, BE101, BE104, BE106, BE109.

KE04 was a quiet student, who did not respond in class unless asked a direct question and only used the computer on one occasion. He is considered an average student by his teacher, but is confidently expected to pass his A-level examination. On the data collected there was no obvious reason why he should do worse than other students in the class who were considered to be weaker at mathematics.

The students BE101, BE104, BE106, BE109 are more interesting. Amongst them are the three poorest marks in their class at calculating numerical gradients from a picture (BE101, BE104, BE109), yet all four were among the eight students in the class who obtained full marks for algorithmic differentiation on both pre- and post-test. Perhaps these are examples of students whose algebraic ability is strong but spatial ability is weak. Insufficient data was collected to test this hypothesis, but it is an interesting phenomenon to study at another time.

Other experimental students in this class are the whole of the group at Cricklade College, who used the computer, yet averaged 1.08 out of 20 on the derivative sketching. Only 4 of the 51 students in the sample correctly selected the graph with a given derivative graph, when a random selection would have given a chance of one in three. None of the four students gave a satisfactory reason for their choice and more than half of the students chose the distractor graph which had the same shape as the given derivative.

As an open access college, there are certainly more weaker students here than the other schools considered, but there is also a sub-population typical of the other sixth-form colleges. One might conjecture that weaker students are significantly less able to cope with the generic organisers in Graphic Calculus than more able students. Other evidence, reported in an open lecture given at a Joint Mathematical Council Meeting on Computers in the Sixth-Form refutes this argument. John Higgo, teaching the lower third ability range at his school taking calculus at O-level, demonstrated that his students were well able to take this kind of derivative sketching in their stride with more complicated examples than those in the post-test.

The performance at Cricklade must be considered in terms of the reports given in the last chapter. The students were given a single lecture in a large group, with no homework set, and then went on to use the computer once which the majority considered to be "too little" or "far too little". There is little evidence that the meaning of the processes demonstrated by the program was explained in any detail by the teachers, on the contrary, there were several students reporting confusion and lack of explanation. Without an appropriate environment the programs are clearly of little value. In the terms of chapter 3, a generic organiser is unsatisfactory without an adequate organisational agent.

## Summary

The experimental and control groups come from two schools which have students of average, or above average ability. A matching of pairs on one or two selected items on the pre-test was made in which the control performed as well as, or better than, the experimental students. This was checked for validity by comparison for bias on other pre-test tasks.

On formal differentiation and the manipulation of the algebra for the gradient of a chord and the gradient of a tangent there is no significant difference between the groups.

On numerical calculation of gradients from a picture with equal x- and y-scales the experimental students with previous calculus experience show an improvement from pre to post test in performance over the corresponding controls, significant at the 1% level using the Wilcoxon test. However, the improvement in the small group of experimental students without previous experience is not statistically significant. The test occurs in a context where one of the items provoked a conflict showing almost random improvement and deterioration from pre- to post-test. It would not be wise to place too much reliance on this result.

The experimental students show significant improvements over the controls on sketching the derivative for a given graph, recognising a derivative and specifying a non-differentiable function. On the three derivative sketching exercises where the formula for the graph was less easy to guess, the significance using a Wilcoxon test was at the 0.1% level. The improvement in the ability to visualise the global derivative produces a performance comparable with that of high ability students entering university to read mathematics.

## 10. Analysis of responses: Open ended questions

The questions analysed in this chapter are open-ended and were given to discover possible differences caused by the approach using the computer compared with teaching without it. Four responses will be analysed:

(G) Differentiation "from first principles"

(H) The gradient of a graph

(I) The tangent to a graph

(J) The derivative of a function.

A fifth item on the post-test concerned the Leibniz notation. The original hope of discussing this in full during the teaching of the topic was not fulfilled. Although this item revealed interesting information, this does not relate to the difference between experimental and control groups, so it will not be considered.

On the four topics mentioned above, two main differences may be hypothesized in advance, caused by the higher level of discussion using the generic organisers and the dynamic computer graphics:

(i) The experimental students may respond to these questions more often than the control students,

(ii) The experimental students may give more explanations of a "dynamic" or "pre-dynamic" kind.

Of these hypotheses, the first is easy to test, but the second depends on the classification of the responses. We will perform the latter in some detail to test the hypothesis and to seek other factors that may arise in the course of the experiment.

During the analysis students will be referred to by number, the prefixes KE, BE1, BE2 being experimental groups, KC, BE1, BE2, BE3, BE4 being control groups. Students with previous calculus experience will be marked with an asterisk, so that, for example, BE108* is the eighth student in BE1, with previous calculus experience. Where appropriate, reference will also be made to the Cricklade students, CE, and those at university, U, who were given the post-test only.

## Interpretation of open-ended questions

The responses for each question will be surveyed and a classification proposed, which attempts to use subcategories of the "dynamic" and "static" types, as in Cornu [1981,1983] and Robert [1982].

Although this proves possible to a certain extent, the various questions on different topics are likely to evoke different kinds of response according to the context. The mental imagery of the mathematical concepts is far more complex than a simple dynamic/static development. There are many occasions in which a dynamic process leads to a static result. For example, the gradient of a graph may be visualized dynamically glancing along the curve, noting the changing gradient and sketching the graph, or it may be seen statically as the global gradient function. When one looks at the gradient graph one may see it dynamically as a process or statically as a fixed global picture, whichever seems more appropriate at the time.

During the development of the calculus dynamic and static concepts are often juxtaposed. For example, the notion of gradient of a line has dynamic elements (e.g. rate of change) as well as static elements (e.g. the fixed gradient of the line). When the notion of gradient of a tangent is introduced through a moving chord, the limiting process is dynamic, yet the tangent at a fixed point is static. The algebraic description of the gradient of the chord tending to the gradient of the tangent has a dynamic feeling to it, but produces a static formula. Likewise the formal calculation of the derivative $nx^{n-1}$ from the formula $x^n$, though a mathematical *process* involving an action, may be seen as simply producing a static formula once more. Having drawn the tangent at the point, one may well see this either as a specific tangent, or a generic idea representing the tangent at an arbitrary point on the graph. One may oscillate from one to the other in a flash, perhaps distinguishing between the ideas at one moment and not at another. It is for this reason that we introduced the term "concept image". The idea of a concept image being evoked in different ways in different contexts may be a helpful formulation to describe what is going on.

Task (G): Calculating the gradient of the tangent "from first principles"

Question 2 on the pre- and post-test is displayed in figure 9.18 in the previous chapter. It exhibits the picture of the graph $y=x^2$, requests the gradient of the chord through (1,1), $(k,k^2)$ and the gradient of the tangent at (1,1). The final part seeks an explanation of how the gradient of the tangent could be calculated "from first principles".

The purpose of this last part was, first, to see if anyone who obtained the expression for the gradient of the chord would let k tend to 1 to get the gradient of the tangent. Secondly, it would be of interest to see if there was a difference between the responses of those who used the computer and those who did not. For example, one might expect more explanations involving the limiting process from those who had used the computer and seen this carried out in a dynamic way. However, there might be peculiarities in the manner in which this is described, due to the nature of the computer representation. For instance, the computer gives a discrete sequence of numerical values tending to a limit, not the limit of a formula.

The responses were analysed with the intention of seeking a classification after the style of Robert or Cornu described in Chapter 2. However, one must note that none of these students (apart from perhaps a few individuals at university) would be likely to know the formal epsilon-delta definition. Instead one may expect one or more of the following:

(1) an explanation of the meaning of the gradient of a line

(2) An explanation involving taking points "very close"

(3) some kind of dynamic limit explanation

(4) An explanation using the calculus.

Response (1) might be expected before exposure to the calculus, with response (4) coming after. Both of these responses are essentially *static*. Responses (2) and (3) may be considered *pre-dynamic* and *dynamic* respectively.

A close study of responses suggests classification into a number of categories. The first four are "dynamic":

<u>DK: dynamic $(k \to 1)$</u>

In this category are placed responses which give correct answers to the first two parts of the question and interpret them in a meaningful way by allowing k to tend to 1. For example:

> Consider the gradient of the line thro A & B as k gets nearer and nearer to 1. (U09)

DF: dynamic limit of the formula $y=x^2$ with $\delta x$ or h increment

This category consists of responses reproducing the formal argument for the formula $y=x^2$. For instance:

$y=x^2$

$\therefore\ y+\delta y=(x+\delta x)^2$

$\therefore\ y+\delta y=x^2+2\delta x x+(\delta x)^2$

$\therefore\ \delta y=2\delta x x+(\delta x)^2$

divide by $\delta x$ & tend $\delta x$ to 0.

$\therefore\ \frac{dy}{dx} = 2x.$ (BC412)

$\frac{(x+\delta x)^2-x^2}{x+dx-x}$

in limit $\delta x \to 0$

$\therefore$ gradient can be found. (BE205*)

DG: dynamic, general argument

A similar category to the previous one, giving a more general description of the limiting argument:

To do this you have to add $\delta x$ to all x, and $\delta y$ to all y then work out, and get limit as $\delta x \to 0$. (BE112*)

DC: dynamic, chord tends to tangent

These responses give a verbal description of the changing gradient of the chord as the limit is taken:

> Take the gradient of the line AB as B gets closer to A, and find the limit. (KE16*)

> Take a small increment of A and test the gradient of the line between A and the increment as the increment tends towards 0. (BC101*)

DN: dynamic "numerical limit" argument

Responses in this category give a limiting argument but mention numbers rather than formulae:

> find the gradient of the line (1,1) to (1.1,1.1) and gradually move these together, noting the value the gradient tends to. (CE20)

The next three categories are not truly dynamic, but begin in the same way as formal dynamic responses; they are coded with a lower case d rather than a capital D.

dS: dynamic substitution

Responses in this category follow the same pattern as the formal limit, but the last stage is accomplished by substituting zero for the increment, without mentioning a limit argument.

$A=(x,y)$

$B=(x+\delta x,y+\delta y)$

$y+\delta y=x^2+2x\delta x+\delta x^2$

$\delta y=x^2+2x\delta x+\delta x^2-y$

$\delta y=2x\delta x+\delta x^2$

$\frac{\delta y}{\delta x} = 2x+\delta x$

At A $\delta x=0$

$\frac{\delta y}{\delta x} = 2x$ $x=1$ gradient=2 (BC306*)

dW: dynamic, partial explanation without limit

Responses in this category begin in the same way as a formal response but stop short of explaining the limiting case.

You could investigate the gradient between a known point A $(x,y)$ and an unknown point B $(x+h,(x+h)^2)$. (BE110*) dE: dynamic explanation, with error(s)

Responses in this category mention $\delta x$ or dx and start off as if to give a formal explanation, but commit an error before completion.

> have $\delta y$ and $\delta x$ and get the gradient
> by $\frac{\delta y-y}{\delta x-x}$ where $\delta y=(x+\delta x)^2$ etc. (BE113*)

Two distinct "pre-dynamic" categories were considered:

VC: "very close", without limiting argument

Here the gradient is calculated by taking two points very close together but without any indication of movement or a limiting argument.

> Taking x & $\delta x$ as points very close together & likewise for y & $\delta y$ then finding the gradient using
> $\frac{y-y_1}{x-x_1} = m$ where m=gradient. (BE107*)

NC: "numerically close" for a specific value

(In the sample, there were no responses in category NC.)

Static responses were classified as:

LG: line gradient formula

These simply explain how to find the gradient of a straight line.

> Find the y-distance between two points on the line AT and divide by the x-distance between the same two points on the line, thus giving the gradient. (KE01)

$$\text{GRADIENT} = \frac{\text{VERTICAL DISTANCE}}{\text{HORIZONTAL DISTANCE}} = \frac{4}{2} = 2. \qquad \text{(KE13)}.$$

CF: calculus formula

The gradient of the tangent is explained as being found by differentiation. (All responses mentioning differentiation were placed in this category.)

> dy/dx=2x ∴ (CE23) [*sic*]

> differentiate $y=x^2$ and substitute value of 1 in for x. (U24)

A few responses did not lie in any of the above categories and were assigned to:

O: other (without dynamic or limit argument)

1/- Take a tangent of point B and a tangent of point A as vector lines.

2/- Take the sum of the vectors to be a vector 'x'

3/- Take 'x' to be the average gradient. (KE14)

The gradient of AT is the gradient of $y=x^2$ at the point (1,1). (BC210*)

Finally the category "nr" denoted "no response".

The full analysis of responses is given in Table 10.1.

The categorization performed by the researcher was checked independently by a second observer. The small percentage of differences (usually due to overlapping of categories) were resolved by agreement.

DK: dynamic, $(k \to 1)$
DF: dynamic, formula $y=x^2$
DG: dynamic, general
DC: dynamic, chord tends to tangent
DN: dynamic, "numerical limit"
dS: dynamic, substitution
dW: dynamic, partial, without limit
dE: dynamic, with errors
VC: "very close", without limiting argument
NC: "numerically close" for a specific value
LG: line gradient formula
CF: calculus formula
O: other

| | DK | DF | DG | DC | DN | dS | dW | dE | VC | NC | LG | CF | O | nr |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Experimental (without previous calculus) (N=12) | | | | | | | | | | | | | | |
| Pre | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 7 | 0 | 1 | 4 |
| Post | 0 | 3 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 3 | 0 | 0 | 2 |
| Control (without previous calculus) (N=15) | | | | | | | | | | | | | | |
| Pre | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 9 | 0 | 0 | 6 |
| Post | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 0 | 0 | 3 | 0 | 0 | 9 |
| Cricklade (without previous calculus) (N=37) | | | | | | | | | | | | | | |
| Pre | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 10 | 0 | 2 | 25 |
| Post | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 4 | 0 | 0 | 0 | 3 | 0 | 29 |
| Experimental (with previous calculus (N=30) | | | | | | | | | | | | | | |
| Pre | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 12 | 9 | 0 | 7 |
| Post | 0 | 3 | 2 | 3 | 0 | 1 | 3 | 2 | 4 | 0 | 7 | 1 | 0 | 4 |
| Control (with previous calculus) (N=52) | | | | | | | | | | | | | | |
| Pre | 0 | 3 | 3 | 1 | 0 | 0 | 0 | 0 | 2 | 0 | 11 | 11 | 3 | 20 |
| Post | 0 | 7 | 2 | 2 | 0 | 1 | 1 | 4 | 1 | 0 | 7 | 5 | 0 | 22 |
| Cricklade (with previous calculus) (N=14) | | | | | | | | | | | | | | |
| Pre | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 4 | 1 | 1 | 7 |
| Post | 0 | 3 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 2 | 0 | 4 |
| University (N=44) | | | | | | | | | | | | | | |
| Post | 1 | 12 | 0 | 1 | 0 | 1 | 0 | 4 | 0 | 0 | 4 | 10 | 0 | 11 |

Table 10.1

Notice that on the pre-test, of the 59 students without previous calculus experience, *none* responded with a dynamic or pre-dynamic argument. This evidence does not support the idea that the notion of a chord approaching a tangent is an "intuitive" method of finding the tangent gradient, in the sense that it is an a *priori* spontaneous method of solution.

It may well be that the phrase "from first principles" intimated unknown technicalities that prevented the evocation of this idea. However, note how many of the 160 students before, and 204 students after, evoked the idea of $k \to 1$. Precisely *one* each time. Again there may have been coercive factors: "k" might be thought of as a constant and not be seen to move, or k may be given a numerical value and again seem fixed. But there were 60 students on the pre-test and 91 on the post-test who obtained the answer $k+1$ or $(k^2-1)/(k-1)$ on the first part, and *only one* in each case evoked the idea of k tending to 1 to obtain the gradient of the tangent.

A more likely explanation is that the students see the description "from first principles" as requiring a particular "approved" response, hence the responses in categories DF, DG, DC and even in LG and CF. The favoured "approved" response is a cultural one, based on the "delta" formulation of Woodhouse [1803].

Despite the numerical flavour of the computer programs, there are only three responses describing a numerical limiting process in category DN and none in the "numerically close" category NC.

If the results are combined into three main classes:

D: those with a dynamic or pre-dynamic explanation

S: those with a static explanation LG, CF or other, O

nr: those with no response

then this gives the breakdown in table 10.2:

| | D | S | nr |
|---|---|---|---|
| **Experimental (without previous calculus)** | | | |
| Pre | 0 | 8 | 4 |
| Post | 7 | 3 | 2 |
| **Control (without previous calculus)** | | | |
| Pre | 0 | 9 | 6 |
| Post | 3 | 3 | 9 |
| **Experimental (with previous calculus)** | | | |
| Pre | 2 | 21 | 7 |
| Post | 18 | 8 | 4 |
| **Control (with previous calculus)** | | | |
| Pre | 9 | 25 | 20 |
| Post | 19 | 12 | 22 |
| **University** | | | |
| | 19 | 14 | 11 |

Table 10.2

In all cases the number responding in the category D *increases* and the number in S *decreases* from pre to post test. However, those not responding *decrease* in the experimental groups and *increase* in the control groups.

Furthermore, if one combines the static and nil responses to give a two way split dynamic/non-dynamic, one obtains table 10.3:

| | D | S/nr |
|---|---|---|
| **Experimental (without previous calculus)** | | |
| Pre | 0 | 12 |
| Post | 7 | 5 |
| **Control (without previous calculus)** | | |
| Pre | 0 | 15 |
| Post | 3 | 12 |
| **Experimental (with previous calculus)** | | |
| Pre | 2 | 28 |
| Post | 18 | 12 |
| **Control (with previous calculus)** | | |
| Pre | 9 | 45 |
| Post | 19 | 32 |

Table 10.3

The pre-test results for the groups without previous calculus experience show no dynamic responses. The pre-test results for those with previous calculus experience show more dynamic responses from the control students. But the post-test show more dynamic responses in the experimental groups in both cases. Applying a one-tail $X^2$ test with null hypothesis that the experimental students are equal or worse than the control students in giving dynamic responses on the post-test gives $X^2=2.72$ (with Yates correction) for those without previous calculus experience; this would (just) be significant at the 5% level. The same $X^2$ test applied to those with previous calculus experience gives $X^2=3.98$, which is also significant at the 5% level.

Thus there are indications that there may be a tendency for the experimental students to give more dynamic responses than the control students. However, this is with uncontrolled groups and should be treated with caution. More reliable indications may be found from the matched pairs.

**Comparison of matched pairs on explaining differentiation from first principles (Task (G))**

We may expect the experimental students to give more responses and more dynamic explanations than the controls. Table 10.2 shows that the experimental and control groups with previous calculus experience differ somewhat on the pre-test. It is wise to look at the selected students in the matched pairs to see if the difference is significant. Counting only those students matched in pairs, and re-arranging the tables into pre-test and post-test, gives table 10.4 for distributions of responses/no responses.

| Response | | nr |
|---|---|---|
| **Pre-test (without previous calculus)** | | |
| Experimental | 8 | 4 |
| Control | 9 | 3 |
| **Post-test (without previous calculus)** | | |
| Experimental | 9 | 3 |
| Control | 7 | 5 |
| **Pre-test (with previous calculus)** | | |
| Experimental | 20 | 7 |
| Control | 18 | 9 |
| **Post-test (with previous calculus)** | | |
| Experimental | 23 | 4 |
| Control | 16 | 11 |

Table 10.4

Here the control students with calculus respond *less* on the pre-test than the corresponding experimental students. Although the post-test for these students shows a marginal increase in the responses of the experimental students and a marginal decrease for the control students, these are clearly not statistically significant.

The students with previous calculus experience differ slightly on the pre-test. Testing with the null hypothesis that the responses of the experimental group on this test are equal to the control group gives $X^2=0.09$, which is statistically (very) insignificant ($0.8<p<0.9$). However, the fact that the control start out with slightly better performances than the experimental group is a warning that one should look carefully at marginal statistical results to see if this factor could affect the balance.

On the post-test, with the null hypothesis that the control students respond equally or more than the experimental students, one obtains $\chi^2=3.32$, which is significant at the 5% level. One should regard this "significance" with a little suspicion as the experimental students started off with a marginal advantage with 2 more responses. For, if one changed the responses slightly, say moving 2 from the control "no response" category to the "response" category, then this would change the result to $\chi^2=1.62$, which would be no longer significant at this level. We therefore suggest that this result is read as suggesting a trend in the right direction but not sufficient to be sure of its statistical significance. In all future references to the level of response, a similar adjustment will be made.

A recount of the dynamic/pre-dynamic responses against the static/no response categories for the matched subsets is given in table 10.5.

| | D | S/nr |
|---|---|---|
| **Pre-test (without previous calculus)** | | |
| Experimental | 0 | 12 |
| Control | 0 | 12 |
| **Post-test (without previous calculus)** | | |
| Experimental | 7 | 5 |
| Control | 3 | 9 |
| **Pre-test (with previous calculus)** | | |
| Experimental | 2 | 25 |
| Control | 5 | 22 |
| **Post-test (with previous calculus)** | | |
| Experimental | 17 | 10 |
| Control | 9 | 18 |

Table 10.5

Here those without previous calculus have no responses in category D on the pre-test, so the responses on the post-test may be compared to test for the improvement of the experimental students. With the null hypothesis that the control students have a greater or equal number of D responses on the post-test one obtains $X^2=1.54$, which is not significant ($0.1<p<0.15$). (This could be due to the small sample size. If the experiment were duplicated with twice the number of students in the same proportions, one would get $X^2=4.2$, giving statistical significance at the 5% level.) Given the evidence one may say that in this sample the experimental students improved more than the control students from pre to post test, but that the changes were not statistically significant.

In the case of those with previous calculus, note that the control students give *more* dynamic responses than the experimental students.Thus it is legitimate to compare the responses on the post-test, with the null hypothesis that the control students have a greater or equal number of D responses. This gives $X^2=3.63$, which is significant at the 5% level.

To test the change in the *quality* of the responses, they were ordinally ranked as follows:

| | |
|---|---|
| nr,0 | 0 |
| LG | 1 |
| CF | 2 |
| dS,dW,dE | 3 |
| VC,NC | 4 |
| DN | 5 |
| DK,DF,DG,DC | 6 |

The marks given to nr, 0, LG and CF are fairly straightforward: the line gradient LG and calculus formula CF are both static responses and are given low marks. The next category is a little more contentious, for dS,dW and dE denote attempts at a dynamic response which may be more erroneous than LG or CF. Nevertheless they do indicate some kind of effort to move in the desired direction and are marked accordingly. The two categories VC and NC show a move to a pre-dynamic response whilst the category DN gives a definite dynamic explanation, still linked to numbers. The four categories DC,DG,DF,DK are all full dynamic responses. The mark scheme is so constructed that a move from any static response to full dynamic gives at least 4 extra marks, whereas any move to the erroneous dynamic category dE is given a maximum of 3. Thus, when *changes* are considered, any change to dE will always rank lower than any change from static to full dynamic.

Using the Wilcoxon test with these scales gives the following statistics:

<u>Without previous calculus</u>

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| pre-test : | −1 | −1 | 0 | 0 | +1 | 0 | 0 | 0 | 0 | 0 | 0 | −1 |
| post-test : | −1 | +1 | +6 | −1 | −3 | +1 | +6 | 0 | +1 | −1 | +6 | 0 |
| improvement: | 0 | +2 | +6 | −1 | −4 | +1 | +6 | 0 | +1 | −1 | +6 | −1 |

We have already checked in chapter 9 that the pre-test result is not significant. Postulating a better performance for the experimental students, with $H_0$: $\Sigma P \leq \Sigma N$, for the post-test and the improvement we have the following Wilcoxon statistics:

post-test : $N=10$, $T=17.5$ : accept null-hypothesis ($p \doteq 0.15$)

improvement: $N=10$, $T=16$ : accept null-hypothesis ($p \doteq 0.12$).

There is an improvement but it is not statistically significant in either case.

Those students who have met calculus before give the following statistics:

With previous calculus

pre-test : +2 +1 −1 +2 −5 −6 +1 −1 −2 −3 +2 −1 +2 −2 +4 0 +1 −6 −1 +2 0 −6 +2 +1 −1 +5 +1

post-test : +3 0 +2 −4 +1 0 +2 −2 −1 −5 +6 +2 +6 0 +4 0 +3 0 −1 +6 +1 −3 +3 +1 +5 +3 −2

improvement: +1 −1 +3 −6 +6 +6 +1 −1 +1 −2 +4 +3 +4 +2 0 0 +2 +6 0 +4 +1 +3 +1 0 +6 −2 −3

post-test : N=22, T=70 : reject null-hypothesis ($p<0.05$).

improvement: N=21, T=56.5 : reject null-hypothesis ($p<0.025$).

Thus both the final scores and the change in scores show a statistically significant improvement.

Task (H): Explaining the gradient of a graph

The last four questions on the post-test requested the students to explain (or "say what is meant by") several key terms that they had been using in the calculus. The first was as follows:

> You have been asked by a student who understands the notion of the gradient of a straight line to explain what is meant by the gradient of a more general graph. Give a brief explanation.

66% of all students responded (including those at university) and 23% mentioned the tangent. Some were quite precise, others less so:

> The gradient is the gradient of the tangent to the curve at any one point. (BC101*)

> It is equal to the tangent at that point. (CE09)

The limiting process was mentioned by only 8 university students out of 44 (18%), 2 experimental out of 42 (5%), and none of the 67 control students. Thus the culturally approved approach to the gradient of the graph is only reproduced by 10 students (5% of the total population).

Looking back to the earlier question on first principles, 14 university students (32%) responded there in terms of limiting processes (categories DK,DF,DG,DC,DN), together with 14 experimental students (33%), and 8 control (12%). Thus the low incidence of limiting responses may be a combination of two factors: first, that they may not evoke the limiting idea *per se* and, second, that they may not consider it appropriate in the context of explaining the idea to another student.

A very obvious difference between experimental and control students is that 18 of the 42 experimental students (43%)

mentioned the idea of lines through two points on the graph as compared with only 4 control students (8%). Of the 18 experimental students, 15 of these specified the points should be close, for instance:

> What it is is the gradient of a straight line drawn between two very close points on a graph. (BE112*)

Several of the students mentioning a line through two close points seem to confuse this with the notion of a tangent:

> The gradient of a more general graph is the gradient of the tangent taken between two points on the graph which are fairly close together. (BC114*)

> The gradient of the graph is found by finding the gradient of the tangent to the curve at that point. The tangent is obtained by joining together 2 points on the graph which are very close together. From this tangent the gradient can be calculated in the same way as you would a straight line. (BE107*)

Only 5 of the 42 experimental students mentioned magnification. For example:

> It is a line that passes through two points on the graph that are so close as to be indistinguishable.

If you were to magnify the graph greatly, you would see it as this straight line. (KE16*)

(As the students' most recent work with the computer had emphasized the notion of a chord clicking along the graph, it may be that the memory trace of magnification may now be less prominent and less likely to be evoked: an interesting conjecture to test on a later occasion.)

Other prominent categories of responses included those who mentioned the rate of change (12% of the total), a quotient of lengths (13%) or the derivative (7%), represented respectively by:

The grd is the rate of change of the function of x with respect to x. (BC402*)

The gradient of a graph is the change in the y coordinate divided by the change in the x coordinate. (CE13*)

The gradient of a straight line is constant. y/x remains the same. In a different graph eg $x^3$ when x=1 y=1 and the gradient is 1. when x=2, y=8, the gradient =4. The gradient is not constant and is always changing. Using a tried and tested method called differentiation the gradient of the graph can be found at any value of x. (BC13*)

A small number mentioned infinitesimally close points (2%), and coincident or consecutive points (2%). For example:

> The gradient of any other graph is equiv. to the grad. of a straight line but only over a very small distance - infinitely small in fact. You treat the curve as a set of small straight lines. (U26)

> A line which passes through 2 consectative points on a graph that can be magnified to produce a straight line. (KE12)

> The gradient of a graph at any specific point is the tangent to the graph at that point and it can be obtained by drawing a chord through two points infinately close together at that point. (KE05)

12% of those responding were not classified in the above categories. Some simply used alternative phraseology, some attempted an explanation, perhaps by stating properties of the gradient:

> The slope of a line on that graph. (CE17)

> A gradient of a more general graph is a steady increase of the y value with the x value. (BC219*)

> The gradient is the angle at which the line is inclined and

is found by taking $\tan\theta$.

> A curve will have different gradients on either side of its maximum or minimum the gradient will be +ve on one side and -ve on the other (CE19)

> The gradient of a curve is the steepness of which it is rising at a certain point. Curves sloping upwards from left to right have a positive gradient and those sloping down from left to right have a negative gradient. The nearer the curve is to vertical the greater the gradient. (BE109*)

It is noticeable that a number of responses in this "other" category attempt to describe the gradient "globally" rather than just at a point:

> The gradient of a general graph is what the gradient is at various points on it. (CE45)

> The gradient is not like a set number all the way through and has usually got an x in. (BC112)

By contrast, the responses outside this category virtually all concentrate on the gradient *at a specific point* and rarely consider the gradient globally. There are some *implicit* references to the gradient at more than one point, usually through the use of the adjective "any" such as the following:

> A gradient of a graph at any point is the gradient of a straight line, just touching that point, called the tangent. (BE205*)

Explicit references to the global nature are rare, and even then, the gradient is not described as a function:

> It is how the gradient changes as if it were made up of an infinite number of straight lines as tangents to an infinite number of points on the graph (CE08*)

> It is the change in y with respect to x at any given point; it can vary from point to point. (BE106*)

Thus, although the experimental students can visualize and *draw* the global derivative as a gradient function, when asked this particular question they tend to evoke an image of the gradient *at a point*. However, it is not easy from a written response alone to glean whether the the tangent is being conceived as being at a *specific* point, or a *typical* (general) point. (One suspects the latter.)

A full breakdown of the responses is given in table 10.6. Entries of the form m+n state that m responses begin in this category and n responses refer to it subsequently. The headings are as follows:

M: magnification
2: gradient of line through 2 points (but not "close")
C: gradient of line through 2 close points
I: gradient of line through infinitesimally close points
K: gradient of line through coincident/consecutive points
L: limiting explanation
T: gradient of tangent
Q: quotient of lengths
R: rate of change, change of x with respect to y
F: calculus formula
O: other
nr: no response

Explaining the gradient of a graph

| | M | 2 | C | I | K | L | T | Q | R | F | O | nr |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Experimental (without previous calculus) (N=12) | 1+2 | 1 | 2+3 | 0+2 | 1 | 0+1 | 5 | 0 | 0 | 0+1 | 1 | 1 |
| Control (without previous calculus) (N=15) | 0 | 0 | 0 | 0 | 0 | 0 | 2+1 | 2 | 1 | 0 | 3 | 7 |
| Experimental (with previous calculus) (N=30) | 1+1 | 1+1 | 6+4 | 0 | 0+1 | 0+1 | 10 | 2 | 4+2 | 2+2 | 3 | 1 |
| Control (with previous calculus) (N=52) | 0 | 1 | 2+1 | 0 | 0 | 0 | 10 | 4+1 | 6+1 | 4 | 9 | 16 |
| Cricklade (without previous calculus) (N=37) | 0 | 1 | 0 | 0 | 0 | 0 | 4 | 2 | 0 | 0 | 4 | 26 |
| Cricklade (with previous calculus) (N=14) | 1 | 0 | 0 | 1 | 0 | 0 | 0+1 | 4 | 0 | 0 | 1 | 7 |
| University (with calculus) (N=44) | 2 | 0 | 3 | 2 | 0+3 | 5+3 | 14+1 | 0 | 4 | 0+1 | 3 | 11 |

Table 10.6

Two factors stand out in this table:

(i) There are more "no responses" among the controls than the experimental students.

(ii) There are more responses in the categories M,2,C,I,K,L among the experimental students than among the controls.

Comparison of matched pairs on explaining the gradient (Task (H))

Performing a recount on the students selected as matched pairs gives the following tables. First those giving responses against nil responses (tables 10.7, 10.8) and then those including responses M2CIKL against those without (tables 10.9, 10.10).

Without previous calculus:

| | response | no response |
|---|---|---|
| Experimental | 11 | 1 |
| Control | 6 | 6 |

Table 10.7

With previous calculus:

| | response | no response |
|---|---|---|
| Experimental | 26 | 1 |
| Control | 16 | 11 |

Table 10.8

Without previous calculus:

| | M2CIKL | other/nr |
|---|---|---|
| Experimental | 8 | 4 |
| Control | 0 | 12 |

Table 10.9

With previous calculus:

| | M2CIKL | other/nr |
|---|---|---|
| Experimental | 12 | 15 |
| Control | 2 | 25 |

Table 10.10

One may ask "what is the probability of such extreme results happening by chance?" For each table, we use a one-tailed $X^2$ test (with Yates' continuity correction) using the null hypothesis that the control students perform as well or better in the first category of each table.

Table 10.7 gives $X^2=3.23$, significant at the 5% level.

Table 10.8 gives $X^2=8.68$, significant at the 0.5% level. Recall that there were two more responses from the experimental than from the control on the pre-test question 2. We may correct for possible bias by checking what would happen if two entries in the control "nr" category are moved to the "response" category. Replacing table 10.8 second line by 18 "responses", 9 "nr" gives $X^2=6.01$, which is still significant at the 1% level. We will take the latter as a safer estimate.

Table 10.9 gives $X^2=9.19$, significant at the 0.1% level. With small expected values, it is wise to check with the Fisher Exact Test, which suggests significance at the 0.5% level.

Table 10.10 gives $X^2=7.81$, significant at the 1% level.

Thus we find on this question:

(i) that experimental students are more likely to respond than control students,

(ii) that experimental students are more likely to give a dynamic, or pre-dynamic response than control students.

Task (I): The notion of a tangent

The next question requested the students to:

> Say what is meant by a tangent to a graph.

The notion of a tangent which is part of our mathematical culture is that of a straight line that *touches* a curve. To this idea is often added the embellishment that it touches the curve *at one point only* and *does not cross it*.

80% of the students responded to this question. 42% mentioned that the tangent touches the curve, 22% intimated that it touches at one point, and 9% specifically stated that it does not cross (or does not cut):

> It is a line touching a curve (BC403*)

> A tangent is a straight line touching only one point on the graph (BC311*)

A straight line which touches but does not cut the graph. (BC216*)

Some responses intimate "touching at one point" or "not crossing" such as:

Touches the graph on the outside. (CE16)

This is a straight line which touches the convex part of the graph. (CE45)

These are counted as "touching", but excluded from the categories "touching once" and "not crossing" because the references are not explicit. Also excluded from the count of "touching once" are those such as:

A line which touches a point on a graph once locally (KE12)

which clearly cannot have the misconception that the graph can touch only once. (Although this response shows greater insight into the possibilities, it is still in error at the highest level of subtlety, as is shown by the counterexample $y=x^2(\sin(1/x)-1)$ at the origin.)

When we turn to responses corresponding to those of categories M,2,C,I,K,L which suggest a trend from pre-dynamic to dynamic limit, we find only one mentioning magnification (category M) and none mentioning a line through 2 points which do not also include some reference to categories C,I,K,L. There are 18 responses (9% of the total) who speak of a line through two very close points (category C), two (1%) mentioning infinitesimally close points (category I) and 12 (6%) referring to coincident points (category K):

> A tangent is a straight line joining two very close points on a curve. (BC314)

> A tangent to a graph may be thought of as a straight line through two points on the graph, an infinitesimal distance apart. (BE210*)

> A line through 2 coincident points on the graph. (BC404*)

Categories C,I seemed more likely to occur amongst the experimental students, but it was noticeable here, and in other places in the test papers, that group BC4 had discussed the tangent in terms of a chord through coincident points. Hence they produced a small number of students giving this traditionally acceptable response.

Very few students chose to use a limiting notion to explain the tangent in this context and these did not give an explanation of the limit in formal terms.

The tangent of a graph is the straight line that gives the gradient at that point as the change in x through the two points the chord is drawn through tends to zero. (KE02)

When two points on a curve are brought close together, the line joining them is a tangent where they meet. (U34*)

The other large category of responses consisted of the 50 students (24%) relating the tangent to the gradient of the graph in some way. The formal definition of a tangent, which is sometimes taught later in the course, and was mentioned to the experimental group at Kenilworth, is a straight line that

(a) goes through the point on the graph
(b) has the same gradient as the graph (where the latter must be calculated by a limiting process or by differentiation).

15 of the 44 university students mentioned the gradient and 14 of these phrased it in a way which could be construed as giving the formal definition:

It is a straight line which gives the slope of a curve or a line for a point on the curve or line. (U06*)

This is the line with the same gradient as the graph at that point at which the tangent touches the graph. (U08*)

Amongst the 35 sixth-formers mentioning the gradient at some stage, the situation is, naturally, more diffuse. Even though all the students at Kenilworth had been given a printed sheet with the above definition, *none* reproduced the definition in this form, for example:

> The tangent to the curve represents the gradient of the curve at one point. (KE08)

Clearly the concept image generated by their experience of handling the concept was more dominant than the little-used concept definition.

More often than not, the gradient was just mentioned amid other comments, often in an imprecise way, referring, for example, to the gradient *of* a point rather than *at* a point on a graph.

> A tangent to a point on a graph is the straight line of the gradient of that point which just touches the point. (KE09).

Sixteen of the 205 responses were classified as "other" than those already mentioned. These included occasional references to the normal:

> A line which is at right angles to the normal of a graph. (BC316*)

or some kind of response with an implicit, or erroneous reference to the normal:

> A line that is at 90 degrees to the circle that touches the point on the graph. (CE07)

These references to the normal may go back to circle geometry in previous mathematical study where the tangent to a circle is constructed by drawing a line at right angles to the radius.

One student gave a powerful dynamic interpretation of the tangent:

> It is the line of motion that a particle would undergo if it were suddenly to break free of the curve without a change in velocity. (KE16*)

Several of the "other responses" were brief explanations that intimated ideas in previously mentioned categories without making them explicit:

> A line drawn parallel to a point on the graph. (BC111*)

A full breakdown of responses is given in table 10.11 under the following headings:

T: touching the graph
1: (touching) at only one point
X: not crossing

M: magnification
C: through 2 close points
I: through infinitesimally close points
K: through coincident points
L: limiting explanation

G: mentions gradient
CF: calculus formula
R: rate of change
O: other

nr: nor response

Explaining the tangent of a graph

| T | 1 | X | M | C | I | K | L | G | CF | O | nr |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Experimental (without previous calculus) (N=12) | | | | | | | | | | | |
| 3 | 0+1 | 0 | 0+1 | 2 | 1 | 1 | 2 | 1+1 | 0 | 0 | 2 |
| Control (without previous calculus) (N=15) | | | | | | | | | | | |
| 7+1 | 0+6 | 0+2 | 0 | 1 | 0 | 0 | 0 | 2+1 | 0 | 2 | 3 |
| Experimental (with previous calculus) (N=30) | | | | | | | | | | | |
| 14+1 | 0+4 | 0+2 | 0 | 8+1 | 1 | 3+2 | 0+2 | 1+5 | 1 | 2 | 0 |
| Control (with previous calculus) (N=52) | | | | | | | | | | | |
| 25+1 | 3+14 | 0+6 | 0 | 2+1 | 0 | 2+1 | 0 | 8+8 | 0 | 5 | 7 |
| Cricklade (without previous calculus) (N=37) | | | | | | | | | | | |
| 13 | 0+6 | 0+2 | 0 | 0 | 0 | 0 | 0 | 4+2 | 0 | 2 | 17 |
| Cricklade (with previous calculus) (N=14) | | | | | | | | | | | |
| 4 | 0+5 | 0+1 | 0 | 1 | 0 | 0 | 0 | 2 | 0 | 3 | 4 |
| University (with calculus) (N=44) | | | | | | | | | | | |
| 15+3 | 1+5 | 0+5 | 0 | 2+1 | 0 | 0+3 | 5 | 10+5 | 0+1 | 2 | 8 |

Table 10.11

Notice that more of the control students respond to this question than the previous one, and there is clearly little difference between the level of responses of the experimental and controls.

Task (I): Comparison of matched pairs on the notion of a tangent

A recount of the responses of the matched pairs confirms this (tables 10.12, 10.13).

Without previous calculus

| | response | no response |
|---|---|---|
| Experimental | 10 | 2 |
| Control | 9 | 3 |

Table 10.12

With previous calculus

| | response | no response |
|---|---|---|
| Experimental | 27 | 0 |
| Control | 23 | 4 |

Table 10.13

However, there is again a difference in the level of response in categories MCIKL (tables 10.14, 10.15).

Without previous calculus

| | MCIKL | other/nr |
|---|---|---|
| Experimental | 7 | 5 |
| Control | 1 | 11 |

Table 10.14

With previous calculus

| | MCIKL | other/nr |
|---|---|---|
| Experimental | 12 | 15 |
| Control | 2 | 25 |

Table 10.15

Using a one-tail $X^2$ test with the null hypothesis that there are equal or fewer experimental students in the MCIKL category in table 10.14, gives $X^2=4.69$, which is significant at the 5% level. The corresponding result for table 10.15 is $X^2=7.81$, significant at the 1% level.

**Task (J): Explaining the derivative of a function**

The final question on the post-test gave the request:

> Explain what is meant by the *derivative* of a function.

The textbook used at Barton Peverill introduces the derivative as the "gradient function", and the same terminology occurs in the computer program. 71 of the responses (34%) mentioned the gradient, for instance:

> The derivative of a function tells you the gradient at any point along the curve of that function. (BE206)

33 of these (16%) referred to it as a function:

> The derivative of a function is the gradient function. A function which determines the gradient at any point on a graph. (BC408*)

9 students (4%) mentioned "rate of change", only 3 students (1%) mentioned a limiting process (all from those at university) and only one student referred to the tangent.

Of course many students mention some aspect of differentiation, but there are 40 students (20%) who describe the derivative only in these terms.

> It is the result of differentiation. (BC316*)

> derivative = differential coefficient ?? (U38*)

> If $y=x^n$
> then the derivative $\frac{dy}{dx} = nx^{n-1}$ . (U27*)

Four students mention other responses, such as:

> It is the new equation which has been formed from the equation of y equals. (BC102*),

but the other major category consist of 78 students (38%) who did not respond at all.

Table 10.16 give the responses broken down into the following categories

G: mentions the gradient
f: also mentions the gradient as a function
R: rate of change
L: limit
CF: calculus formula
T: tangent
O: other
nr: no response.

Explaining the derivative of a function

| G | f | R | L | CF | T | O | nr |
|---|---|---|---|---|---|---|---|
| Experimental (without previous calculus) (N=12) | | | | | | | |
| 9 | 0+6 | 0 | 0 | 1 | 1 | 0 | 1 |
| Control (without previous calculus) (N=15) | | | | | | | |
| 1 | 0+1 | 0 | 1 | 3 | 1 | 0 | 11 |
| Experimental (with previous calculus) (N=30) | | | | | | | |
| 20 | 0+10 | 1+1 | 0 | 6 | 0 | 1 | 2 |
| Control (with previous calculus) (N=52) | | | | | | | |
| 18 | 0+10 | 2 | 0 | 15 | 0 | 2 | 15 |
| Cricklade (without previous calculus) (N=37) | | | | | | | |
| 3 | 0 | 0 | 0 | 8 | 0 | 0 | 26 |
| Cricklade (with previous calculus) (N=14) | | | | | | | |
| 4 | 0+1 | 0 | 0 | 1 | 0 | 0 | 9 |
| University (with calculus) (N=44) | | | | | | | |
| 15+1 | 0+5 | 5 | 3 | 6 | 0 | 1 | 14 |

Table 10.16

Comparison of matched pairs on the notion of a derivative (Task (J))

Recounting all those in the matched groups who respond, and comparing with those who do not, gives interesting information (Tables 10.17, 10.18):

Without previous calculus

| | response | no response |
|---|---|---|
| Experimental | 11 | 1 |
| Control | 4 | 8 |

Table 10.17

With previous calculus

| | response | no response |
|---|---|---|
| Experimental | 25 | 2 |
| Control | 17 | 10 |

Table 10.18

Using a one-tail test with null hypothesis that the experimental students respond at an equal or lesser level gives $\chi^2=6.40$ for table 10.17, which is significant at the 1% level, and $\chi^2=5.25$ for table 10.18, significant at the 2.5% level. Recalling once more the bias against the control on response to question 2 on the pre-test, one sees what will happen with a move of two responses between the control categories. This still gives $\chi^2=3.06$, which is significant at the 5% level, and we take the latter as a safer estimate.

In these tables, a number of the responses simply refer to the derivative in terms of the process of differentiation, without relating to any other concept. A classification into those who respond using other ideas and those with either no response or CF is given in tables 10.19, 10.20.

Without previous calculus

| | GfRTLO | CF/nr |
|---|---|---|
| Experimental | 10 | 2 |
| Control | 1 | 11 |

Table 10.19

With previous calculus

| | GfRTLO | CF/nr |
|---|---|---|
| Experimental | 19 | 8 |
| Control | 13 | 14 |

Table 10.20

Here a one-tail test $X^2$ with null hypothesis that the experimental students respond at the same or lesser level in categories GfRTLO gives $X^2=10.74$, significant at the 1% level. However, the same test applied to table 10.20 gives $X^2=1.92$, which is *not significant*.

If we look at those students who explain the derivative in terms of the gradient, we obtain tables 10.21 and 10.22, where the numbers given in brackets also refer to the gradient as a function.

Without previous calculus:

| | Gradient (function) | other/nr |
|---|---|---|
| Experimental | 9 (6) | 3 |
| Control | 1 (1) | 11 |

Table 10.21

With previous calculus:

| | Gradient (function) | other/nr |
|---|---|---|
| Experimental | 17 (8) | 10 |
| Control | 10 (9) | 17 |

Table 10.22

Those without previous calculus experience clearly have more responses for the gradient, and here a one-tail $X^2$-test, with the null hypothesis that the control students give as many or more gradient responses than the control, gives $X^2=8.4$ (significant at 1% level). Even looking at those who respond with both gradient and function, compared with those who respond either gradient alone or other/nr, gives $X^2=3.23$, which is significant at the 5% level.

But the difference between the experimental and control students who have had previous calculus experience is not statistically significant. A one-tail $X^2$-test applied to table 10.20, with the null hypothesis that the control students mention the gradient as much, or more than, the experimental students, gives $X^2=2.66$, which is *not* significant at the 5% level.

Interpretation of results

The picture that these results suggest is an interesting one. Tables 10.17 and 10.18 intimate that those doing calculus for the first time following traditional patterns of teaching may not have had the experience to be able to explain the nature of the derivative in any terms other than carrying out the process of differentiation. The combination of the use of the computer and further discussion has a significant effect on students meeting differentiation for the first time, evoking the idea that the derivative is the gradient of the graph. Tables 10.18 and 10.20 show that those control students studying calculus for a second time are beginning to make up the leeway and coming closer to the performance of the experimental students. However, even though the control students use the term "gradient function" here almost at the same level as the experimental students, we showed in chapter 9 that when it comes to interpreting the derivative of a graph as a gradient function by sketching the gradient graph, the experimental students are significantly better.

### The limiting concept

As we have classified the responses to the various open-ended questions, it will not have escaped the reader's notice that there are very few explanations which refer to the limiting concept. If we perform a recount for the four groups: experimental and control, with and without previous calculus (as selected for matched pairs), and the university students, we obtain table 10.23. Here limiting responses in the "first principles" question are taken to be those in categories DK,DF,DG,DC,DN, and for the gradient, tangent and derivative they are those in category L. The asterisks denote the control and experimental groups with previous calculus experience.

Responses mentioning limiting processes

| | Exp (N=12) | Contr (N=12) | Exp* (N=27) | Contr* (N=27) | University (N=44) |
|---|---|---|---|---|---|
| 1st princ. (pre) | 0 | 0 | 1 | 5 | - |
| 1st princ.(post) | 4 | 1 | 6 | 6 | 14 |
| gradient | 1 | 0 | 1 | 0 | 8 |
| tangent | 2 | 0 | 1 | 0 | 5 |
| derivative | 0 | 0 | 0 | 0 | 3 |

Table 10.23

The low level of responses indicates the high cognitive demand of this general concept. Although the notion of a limit is the foundation of the *mathematical* development of the calculus, it is not a natural starting point for the *cognitive* development. As we saw, prior to meeting calculus, none of the students evoked a limiting idea to calculate the gradient of the tangent in terms of the limiting chord, and even after exposure to the calculus, students reproduced the culturally accepted explanations of "first principles" rather than attempt to develop the idea in a relational way. This suggests strongly that one must build up to the notion of limit rather than use it as a cognitive starting point.

The generic organisers help the experimental students develop a global gestalt of the gradient concept, but the full formation of the limit process is still only partially developed in most of the students. What is noticeable also is that there are more responses from the experimental students, of the kind which considers the tangent as a line through two very close points.

This is, of course, mathematically incorrect. However, the culturally embedded notion of a tangent as a line which touches a curve at a single point and does not cross it, is more prevalent amongst the control students, and this also may cause later problems. We shall return to these points in the next chapter.

## Summary

The interpretation of the open-ended questions has proved to be a lengthy and not always easy task.

The significance levels found for the assumption that the experimental students respond more often that the control are as in table 10.24, where the first column refers to those without previous calculus experience and the second to those with ("n.s." denotes "not significant"):

Significance that experimental students *respond more often*

| | without | with |
|---|---|---|
| "First principles" | n.s. | n.s.* |
| gradient of a graph | 5% | 1% |
| tangent to a graph | n.s. | n.s. |
| derivative of a graph | 1% | 5% |

Table 10.24

The method of differentiation from "first principles" does not evoke a significant difference in responses, although there is a tendency for experimental students to respond *more* and the control students to respond *less* from pre-test to post-test. The improvement in the experimental students with previous experience is close to being statistically significant (marked with an asterisk).

The difference on responses to the explanation of the gradient is highly significant, particularly when one notes the small number of students in the category without previous calculus experience, which requires a very skewed response to give statistical significance. Clearly the experimental students are much more willing to respond than the controls in this case.

There is absolutely no difference in the level of response on the tangent question, because the tangent is a culturally embedded concept that most people have a feeling for (though this feeling may not correspond to the formal definition). But when we move on to explaining the derivative of a graph, the experimental students are again more willing to respond. This is not because they are better at differentiation, for the previous chapter showed they were not. As we have seen, their responses indicate it is because they are more likely to relate the derivative to the gradient of the graph.

Looking at the four rows of responses in table 10.24, one comes to an inescapable conclusion that the original hypothesis may be refined to a more precise statement. It is not generally that the experimental students are more likely to respond, but specifically that *the experimental students are more likely to respond to those questions which cause them to evoke the global gestalt of the gradient of the graph.*

This occurs only to a limited extent in the "first principles" question where the dynamic limit is the chord tending to the tangent. Although the computer program includes a dynamic representation of the chord tending to the tangent graphically and numerically, which might cause an improvement in dynamic/pre-dynamic response, this was little used.

The evocation of the global gradient does not seem to be evoked explicitly in the explanation of the gradient of the graph; the responses refer only to the gradient *at a point*. However, they are often in terms of the tangent, or the gradient of a line through close or coincident points, an image given to the experimental students when the computer program draws a chord clicking along the graph through two nearby points. Thus there is the possibility that the explanation of the gradient at a point may be related to a "snapshot" of the computer-drawing process, a "freeze-frame" of the global procedure.

There is no difference on the level of response to the tangent itself, but the response to the derivative is again higher because of the many references to the gradient of the graph.

The major difference between the experimental and control is not just in terms of the *number* of responses, it is also a question of *content*, with a larger number of experimental students giving more dynamic/predynamic responses than the controls (table 10.25).

| Significance that experimental students give more dynamic/pre-dynamic responses | | |
|---|---|---|
| | without | with |
| "First principles" | n.s.* | 5% |
| gradient of a graph | 0.5% | 1% |
| tangent to a graph | 5% | 1% |

Table 10.25

Here there is significance in every case, except those without previous experience of calculus on "first principles". Even here there is a clear tendency for the experimental students to give more dynamic/pre-dynamic responses, but the small sample requires a large bias to produce statistical significance and this is level is not attained.

When we turn to the explanation of the derivative, although a dynamic limit explanation is possible, few students give it. However, the proportion of experimental students giving a response other than a tautological description purely in terms of differentiation is greater than that of the control students. There is a tendency for the experimental students to give more responses describing the derivative as giving the gradient of the graph. However, the control students with previous calculus experience make up the leeway and are just as likely to describe the derivative as the gradient function. The significance of responses is given in table 10.26, where n.s. denotes "not significant" and (in this case) "n.s.*" denotes "not significant at the 5% level, but significant at 10%".

<u>Significance that experimental students are more likely to explain a derivative in the following terms:</u>

| | without | with |
|---|---|---|
| response other than differentiation | 1% | n.s.* |
| gradient | 1% | n.s.* |
| gradient function | 5% | n.s. |

Table 10.26

Remember that these responses should be seen in the light of the previous chapter. Although the control students with previous calculus experience are just as likely as the experimental students to speak of the gradient function, they are less able to sketch the derivative of a function given the graph. One may hypothesise that their response is more instrumental and less related to a richer concept image of the global gradient.

# 11. Concept images of limits and tangents

In the previous two chapters we saw that the teaching using the generic organisers on the computer was capable of giving an improvement in the visualization of the gradient function and more responses of a dynamic/pre-dynamic nature to questions involving limiting processes, tangents and gradients. We also saw that very few of the open-ended responses, amongst the control or experimental students, mentioned limiting processes, suggesting that the high cognitive demand for this general concept renders it an inappropriate starting point for a cognitive development of the calculus. However, the proposed cognitive development using the generic organisers may have what seem to be less desirable side-effects, for example that students using the computer were more liable to say that a tangent was a line through two "very close" points.

In this chapter we shall report evidence that both experimental and control students had more belief in the truth of statements involving the dynamic notion of the extended chord tending to the tangent than with statements mentioning the term "limit". We shall also show that the notion of a tangent being a line through two "very close" points is common amongst the control students.

## The language of the calculus

In chapters 2 and 3 we discussed ways in which the language of calculus contains many terms such as "tends to", "approaches", "converges", "limit", where the vernacular meaning subtly affects the student's interpretation of the mathematical meaning. For instance, there is a wide-spread belief that the terms "tends to" or "approaches" indicate that the limit may not be attained.

An attempt was made to gain some insight into the students' interpretation of the language by asking them to respond on the pre-test and post-test as to whether they considered certain statements to be true or false (on a four-point sliding scale TRUE, true, false, FALSE). Interviews with selected students following the pre-test showed that these questions were too blunt an instrument for their intended purpose, for they failed to pick up the wide spectrum of reasons behind the student responses. For example, a student faced with figure 11.1 and asked to respond to the veracity of the statement:

As $B \to A$ the line through AB tends to the tangent AT

might respond "TRUE".

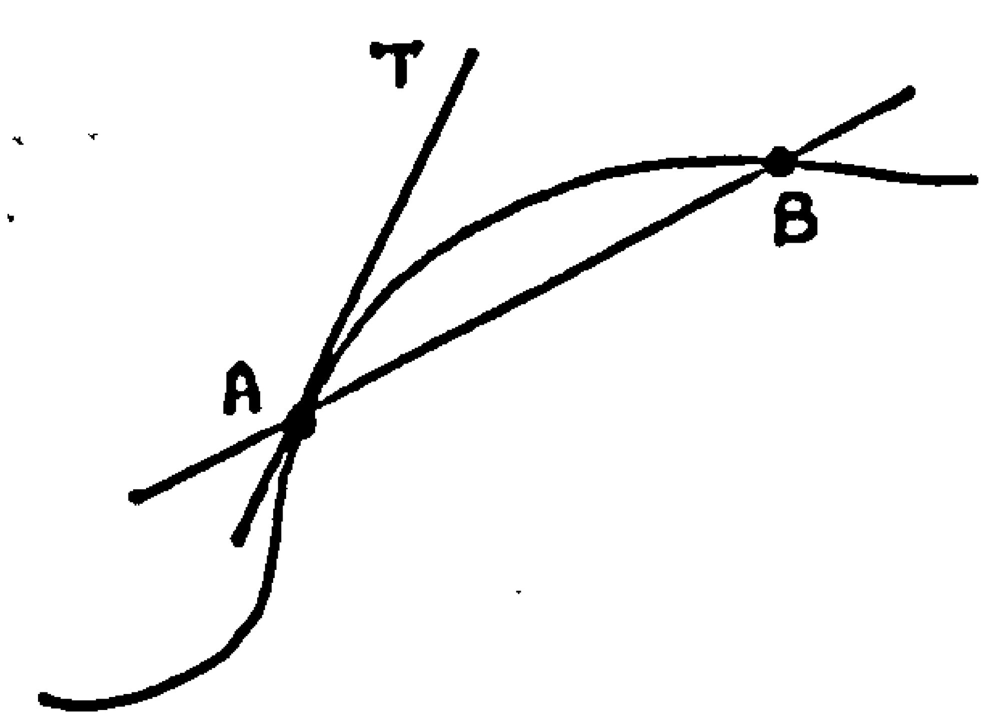



Figure 11.1

But an interview with such a student (KE15*) uncovered the fact that he interpreted the symbol B→A as the notation for a fixed vector direction AB. He believed that B moved to A, *along the chord* and so the line (segment) AB moved closer to A and "tended to the tangent". His "correct" response to the question was based on an incorrect interpretation of what was going on (even though he is a student with previous experience of calculus).

Of nine students interviewed after the pre-test, four saw the point B moving down the chord and responded "true" for an inappropriate reason.

It is also possible for a student to have a good visualization of the process, yet respond that the statement is "false". For example, Adam (KE14) visualized B moving round the curve, as required, but explained that the lines were infinite, and, no matter how close B got to A, way off at infinity the line AB and the tangent AT were still a long way apart...

Another student, Alan (KE07*), who had studied calculus over a year before, elsewhere in the interview insisted that the chord would never reach the tangent for exactly the same reason as Adam, yet, unlike Adam, asserted the given statement to be "TRUE". Thus the classification of the students' responses into a spectrum of "true" or "false" categories, on the pre-test at least, is rendered almost meaningless as it includes students giving the "right" response for a "wrong" reason and others giving the "wrong" response for a perceptive reason. It is made less reliable still by the desire for some students to put down what they think is required by the researcher, following some kind of guessing pattern. As Alan (KE07*) said:

> "Because you put 'true' or 'false' down doesn't mean it's necessarily 'true' or 'false', it's 'cause whatever I feel I *ought* to put down ... well, what looks right. ... If it had a 'true' before I'd put a 'false' down and a 'false' before I'd put a 'true' down."

We will not use this question for a statistical pre-test/post-test comparison. However, taken with the discussions with individual students, it does give some insight on the student's use of language which will be considered in the next section.

"Tends to" and "limit"

Six of the statements used on the pre-test and post-test were descriptions of the limiting process. All of them refer to figure 11.1, where the intention is that the line through AB is to be seen moving round to a tangential position AT. The question was prefaced with the comment:

Referring to the diagram, which of the following statements would you say are true and which are false? If you are ABSOLUTELY SURE, underline the response in CAPITALS, otherwise underline the response in lower case letters. In each statement the "line through two points" or the "tangent" means the whole line, not just the line segment between the two points concerned.

There followed a list of statements from which we have selected the following for our discussion:

(1) As $B \to A$, the line through AB tends to the tangent AT.

(2) As B tends to A, the gradient of the chord AB tends to the gradient of the tangent.

(3) As $B \to A$, the line through AB approaches the tangent as a limit.

(4) As $B \to A$, the line through AB has the tangent AT as a limit.

(5) As $B \to A$, the limit of the gradient of the chord AB is the gradient of the tangent.

(6) $\lim_{\text{as } B \to A} \{\text{gradient of chord AB}\} = \text{gradient of tangent AT}.$

Items (1) and (2) are fairly standard mathematical definitions, the first referring to the picture of the chord moving towards the tangent and the second, in mathematical terms, a more precise statement as it refers to the way the gradient of a chord (a variable number) tends to the gradient of the tangent (a fixed number). Both are strongly dynamic in feeling and neither contains the word "limit".

All the remaining four sentences include a dynamic element as B tends to A, but all include the term "limit". (3) and (4) only differ in that one word "approaches" is exchanged for "has". (5) is a fairly standard limit definition, and (6) is the definition given in the text book used by the Barton Peverill students (Bostock & Chandler [1978]).

The total number of true responses (i.e. "TRUE or true") for each question is given in table 11.2, with the number responding "TRUE" in brackets. Percentages are given below each line in italics.

| | (1) | (2) | (3) | (4) | (5) | (6) |
|---|---|---|---|---|---|---|
| Experimental without previous calculus (N=12) | | | | | | |
| Pre-test | 8(1) | 7(5) | 8(0) | 5(0) | 5(0) | 4(1) |
| % | *67(8)* | *58(42)* | *67(0)* | *42(0)* | *42(0)* | *33(8)* |
| Post-test | 12(9) | 10(7) | 9(5) | 10(8) | 7(3) | 9(6) |
| % | *100(75)* | *83(58)* | *75(42)* | *83(67)* | *58(25)* | *75(50)* |
| Control without previous calculus (N=15) | | | | | | |
| Pre-test | 10(2) | 8(3) | 4(0) | 5(0) | 4(2) | 4(0) |
| % | *67(13)* | *53(20)* | *27(0)* | *33(0)* | *27(13)* | *27(0)* |
| Post-test | 14(9) | 12(6) | 6(0) | 11(4) | 9(0) | 5(0) |
| % | *93(60)* | *80(50)* | *50(0)* | *73(27)* | *60(0)* | *33(0)* |
| Experimental with previous calculus (N=30) | | | | | | |
| Pre-test | 21(14) | 19(9) | 13(2) | 12(6) | 19(6) | 14(6) |
| % | *70(47)* | *63(30)* | *43(7)* | *40(20)* | *63(20)* | *47(20)* |
| Post-test | 27(21) | 28(19) | 16(6) | 23(12) | 24(13) | 24(19) |
| % | *90(70)* | *93(63)* | *53(20)* | *76(40)* | *80(43)* | *80(63)* |
| Control with previous calculus (N=52) | | | | | | |
| Pre-test | 42(17) | 33(16) | 25(7) | 25(11) | 25(12) | 22(9) |
| % | *81(33)* | *64(31)* | *48(13)* | *48(21)* | *48(23)* | *42(17)* |
| Post-test | 49(26) | 42(20) | 38(10) | 39(21) | 35(14) | 36(19) |
| % | *94(50)* | *81(38)* | *73(19)* | *75(40)* | *67(27)* | *69(37)* |
| University [post-test only] (N=44) | | | | | | |
| | 42(36) | 39(30) | 28(10) | 34(17) | 35(9) | 38(30) |
| % | *95(82)* | *89(69)* | *64(23)* | *77(39)* | *80(43)* | *86(69)* |

Table 11.2

As we have explained earlier, these figures are likely to be misleading in that students may give the "right" response for the "wrong" reason, or vice versa. However, we see that in every category those responding "TRUE or true" increase from pre-test to post-test, with the sole exception of question (5) amongst the control without previous calculus. Even here, the "TRUE" category only drops from 2 to 0 whilst the "TRUE or true" reponses increase from 4 to 9.

Statements (1) and (2) always score more "TRUE" or "true" responses than (3) to (6) except for two instances where (2) is marginally below one of the other four. Responses involving the limit concept seem to cause more difficulty to students and this was confirmed by discussion with individuals who often referred to the vernacular meaning. Most had not been given a concept definition and evoked a concept image based on their previous experience.

After the pre-test, Alan (KE07*) said (about question (3)):

> "I didn't understand what it meant by 'as a limit', I mean, I thought maybe that's as far as it would go, but didn't think that was right."

When I queried "Are you using the word 'limit' in the sense of something you can't get past?", he said:

> "I thought it might mean that, but I didn't think it did, I thought it had some other meaning which I didn't know."

Others were very definitely interpreting "limit" as something that couldn't be passed. Andrew (KE02) saw that the extended line AB was unlimited in length, so the tangent AT could not be its limit:

> "I presumed a limit to be an end, taking the limit in its literal sense. The line AB just continues to infinity on

either side of it, so the tangent cuts it at a definite point, a finite point, not an infinite point, which meant that it was not a limit."

Martin (KE06) saw the point B moving down the (fixed) chord and thought the tangent was not a limit for AB as B would eventually pass over it to the other side:

"because the chord keeps on going down there, as B goes along the chord, it would pass over A."

Ian Pringle (KE11) wasn't sure whether the statements referred to the straight line AB, or the part of the curve from A to B. The line AB didn't have the tangent as a limit because part of the line was on the other side of the tangent. But the *curve* AB did have the tangent as a limit, because, near the point of contact, A, curve was all on one side of the tangent:

.If it was the line AB, it hasn't got a limit... The straight line goes through AT, through A. They cross, so it doesn't have a limit, it goes past. [...] Then I thought you could be speaking about the curve, ... it has a limit at the tangent, or at that point of the curve anyway," (pointing at A). "If you see when it goes, it actually comes back round." (He traced the part of the curve on one side of the tangent with his finger as he spoke.)

Gordon (KE12), saw the point B moving round the curve in the conventional sense but initially thought the line through AB couldn't have the tangent as a limit, saying:

> "Limit... that's the thing I wasn't sure about 'cause if it's the limit it's never actually going to get there, so the limit won't actually be AT, will it?"

When I asked, "are you using the word limit in the sense of a speed limit, something you can't pass ?", he went on:

> "As close as you *can* get to it, but you'll never be able to get to AT, will you? ... So AT *couldn't* be the limit, it'd be the ..., very close to AT would be the limit."

He appeared to be articulating a generic limit concept in the sense discussed in chapter 3. I commented, "Do you say that there'd be a limit before you get to AT?"

> "Yeh...," (then he thought a moment), "*There wouldn't actually be ever a limit, would there?* ... Yeh,... you'd get smaller and smaller and smaller, the distance between, the difference, ... wouldn't it?"

Moving on to (3) he again returned to the generic notion that a limit couldn't be attained.

"It does [approach it], but it never gets there, so its always approaching it."

However, as he was explaining why he thought that statement (2) (which describes the gradient of the chord tending to the gradient of the tangent) was "TRUE" whilst (5) (which mentions the limit) was "false", he saw for himself a possible new meaning of the limit process, which is an intuitive version of the formal limit. Of (2) he said:

"Tends to means it gets closer to it, getting nearer and nearer..."

but (5) was "false":

"'cause it will never actually be the tangent... unless you took the limit to be the ..." (He thought for a moment.) "If you rounded it up, once you'd taken it so close, then you rounded it up, then *that* was the limit, then it would be true..."

Although the interpretations just described were held just after the pre-test, general difficulties with the limit concept persisted through to the post-test in the sense that table 11.2 shows that all those items mentioning limits scored fewer true responses than those which did not.

Throughout the pre-test and the post-test, statements (1) and (2), describing the process dynamically without mentioning limits, consistently scored the largest number of "true" responses. Although (2) is a marginally more satisfactory mathematical statement than (1) (dealing with limits of numbers rather than lines), it is a little more complicated and scored lower in the students' estimation. Allowing for a number of possible responses like Alan earlier, who reacted according to how many previous responses had been true or false, both these descriptions meet a with a high level of approval from the students.

A far greater cause for concern is the low number of students feeling fully secure with (4), (5) and (6), which are fairly standard versions relating the limit of the chord gradient to the tangent gradient. On the post-test all student groups averaged less than 50% "absolutely sure" in the belief that (5) was "TRUE". Definition (6), (used by the text-book Bostock and Chandler [1978]) created difficulties partly because of the way it was typed on the test-paper. (It was often read with the symbols "as $B \to A$" on a second line, following the rest of the statement, rather than as a subscript of the limit notation.) Some students were also fooled by the curly brackets, thinking that it had something to do with set-theory.

Even students who have completed their A-level course to a high standard and gone on to read mathematics at university are not immune to insecurities. The last line of table 11.2 shows 82% "absolutely sure" of statement (1) and 69% "absolutely sure" of statements (2) and (6), but statements (4) and (5) drop to 39% and 43% respectively.

This all demonstrates the problems we face when trying to be precise with notation. *It can only be precise to someone who has the sophistication to understand all the terms.* Beginners, who are left to make do with a concept image generated by their previous experience, often interpret mathematical statements in a manner quite different from what is intended.

Tangents and lines through "two very close points" on a graph

In chapter 10 it emerged that the experimental students were more likely to suggest that the tangent was a line through two close points on the graph in response to an open-ended request to say what is meant by a tangent. The apparent inability of students to distinguish between a theoretical tangent and a line through two close points on the graph could be a source of criticism of the computer approach by professional mathematicians.

The list of statements to which the students were asked to respond on the pre-test and post-test included:

The tangent AT is the line through two very close points on the graph

On the pre-test the responses were as in table 11.3 (where "nr" denotes "no response":

| | | TRUE | true | false | FALSE | nr |
|---|---|---|---|---|---|---|
| Without previous calculus experience | | | | | | |
| Experimental | (N=12) | 0 | 5 | 0 | 7 | 0 |
| Control | (N=15) | 0 | 2 | 4 | 8 | 1 |
| With previous calculus experience | | | | | | |
| Experimental | (N=30) | 3 | 4 | 9 | 14 | 0 |
| Control | (N=52) | 13 | 9 | 10 | 20 | 0 |

Table 11.3

Those without previous calculus include a *majority* who were definitely sure that the statement is "FALSE" and *none* thinking it to be "TRUE". But those with calculus experience were already beginning to diversify more in their opinions, with 16 out of 82 (20%) considering it to be "TRUE" and a further 13 (16%) tending to think it true. This shows that a third of the students with previous calculus experience (but no acquaintance with the computer) at least suspected the truth of the statement.

Ian Pringle (KE11) was one of those without prior knowledge of the calculus who was sure it was FALSE:

> "... because we've been taught that it doesn't touch two points, it just scrapes one point, [...] I think our teacher used 'scrapes' so that we can visualize it easier."

(Ian had moved to the school from Scotland, so this particular earlier experience wasn't shared with the other students.)

Talking to some of the other students revealed conflicts in their thinking at this stage. Naomi (KE01) thought the statement to be "true", but remembered that she had been told that a "tangent only meets the graph at one point". She went on to say

> "... a line didn't just touch it and then leave again, it carried on, kind of, along the curve, so it's got to touch that moving point..."

She seemed to visualize a point moving along the curve and was unable to see how a line could touch at one point only without staying in contact a little beyond the point of contact.

Andrew (KE02) also thought it "true", but said:

> "If you draw a tangent, obviously it cuts two very close points of the line, ... I'm not sure whether the points are two very close ones or, as in the next question, ... two coincident ones."

Ian Wells (KE16*), who had previous experience of calculus had a clearer idea, but again with seeds of conflict. He said

> "The definition of a tangent is that it passes through only

one point"

but asserted

"The only way to work out a tangent is to work it out with two points."

This perceptive observation is the root of the problem with tangents. Apart from suggesting how one might sketch a tangent on a picture of a graph, the naive definition is inoperative. It specifies a property of a tangent which is not true in general and does not provide a theoretical method by which a tangent may be constructed. On the other hand, the definition of a line through two close points enables one to calculate a good approximation.

On the post-test the distribution of the student responses changed to those in table 11.4:

| | | TRUE | true | false | FALSE | nr |
|---|---|---|---|---|---|---|
| Without previous calculus experience | | | | | | |
| Experimental | (N=12) | 6 | 5 | 0 | 1 | 0 |
| Control | (N=15) | 2 | 5 | 2 | 6 | 0 |
| With previous calculus experience | | | | | | |
| Experimental | (N=30) | 12 | 7 | 6 | 5 | 0 |
| Control | (N=52) | 13 | 9 | 13 | 16 | 1 |

Table 11.4

Now the *majority* of the experimental students believe that a tangent is a line through two close points on the graph (92% of those without earlier calculus experience, 63% of those with). That this is not due solely to the use of the computer is evidenced by the seven out of 15 control students without previous calculus (47%) and 22 out of 52 with previous calculus (42%) who also believe the statement to be true. The control students are not exempt from the conflict, on the contrary, they give a full spectrum of responses from "TRUE" to "FALSE".

This spectrum of belief remains with those at university (with a bias towards "FALSE"), as in Table 11.5:

| | TRUE | true | false | FALSE | nr |
|---|---|---|---|---|---|
| University [post-test only] (N=44) | 6 | 7 | 9 | 22 | 0 |

Table 11.5

Summary

In common with the work of Cornu [1981,3] we find there are difficulties with the vernacular notion of limit conflicting with the mathematical notion, giving no comfort to those who see the limit concept as the appropriate foundation for the theory of calculus. One may conjecture that a cognitive approach is better served by giving appropriate experiences that will build up to the general limit concept rather than using it as a starting point. However, the cognitive approach using the computer leads to an increasing belief that a tangent is a line through two very close points on the graph. This occurs despite the fact that the computer programs quite clearly label the line through two close points as an extended chord and not as a tangent.

If one is trying to develop a cognitive approach it is essential to acknowledge the realities of student thinking. There is often a profound wisdom in their views which show a subtle alternative way of looking at a problem. The notion of a line through two close points on a graph is an *operative* definition to obtain the gradient. It can be used to find the gradient numerically with a computer or hand calculator and this may prove to be an ideal pre-dynamic concept to lead into the formal definition of a tangent. A *practical* way to find the gradient is to calculate it using two very close points; a *theoretical* way to calculate the gradient precisely at a later stage may be to use this idea as a cognitive foundation for the introduction of the limit concept.

## 12. Concept images of gradients, tangents and derivatives

The traditional approach to teaching the calculus is to use simple examples of tangents and gradients in the first place, delaying any non-examples to a much later stage. It is considered that left and right derivatives are difficult to understand, with their enigmatic limit notations such as

$$\lim_{h \to 0-} (f(x+h)-f(x))/h$$

for the left derivative and

$$\lim_{h \to 0+} (f(x+h)-f(x))/h$$

for the right derivative.

Skemp argues that one cannot understand a concept fully from examples alone, it is also essential to have *non-examples*. In the experimental approach to the calculus used in this thesis, the aim was to give both examples and non-examples at the outset. Thus the notion of gradient is introduced by magnifying the graph and, if its curve highly magnified looks almost straight, then the gradient of the graph is the gradient of this highly magnified straight portion. Conversely we can say that the graph does not have a gradient at a point if it never magnifies to look straight there. For instance, one may give non-examples which have different left and right gradients. These magnify to have a corner. Likewise, with a little ingenuity, it is possible to look at non-examples which are too wrinkled to have even a one-sided gradient.

One may hypothesise that this enables students to develop a more rounded mental image of the concept, knowing not only what it is, but also what it is not.

The purpose of this chapter is to analyse two short investigations given during the period of study to the control and experimental groups, and also to a group of mathematics students at university (appendices 4 and 5). Unlike chapter 9, which considers the *global* gradient function, these investigations study the students' ability to respond to examples and non-examples of gradients and tangents *at a point*. A final question also links these to the notion of derivative.

One may hypothesise that the experimental students are more able to recognise examples and non-examples of gradient and tangent than the control students. It may happen that earlier concept images (for example the naive idea that a tangent "touches at one point only") may interfere with the computer experience, but one may conjecture that the experimental students will be more successful in throwing off the shackles of their earlier naive impressions.

## The Gradient Investigation

The gradient investigation was given during the course of the sixth-form work and was done by the students at Kenilworth and Barton Peverill. The Head of Department at Cricklade College declined to give it to his students and they took no further part in the research.

The paper consisted of six questions, all with the same format, displaying the graph of a named function with the question:

> Can you calculate the gradient at x=0? YES/NO
> If YES, what is the gradient? If NO, why not?

They were then asked:

> Are you sure of your answer?
> Underline the response that best fits your feelings:
> Certain / fairly sure / fairly doubtful / very doubtful.

The intention was to test the students' concept images of the gradient notion in unusual circumstances. In practice the statistical differences on the first part of the question were so great that the final part gave little extra information and will not be used.

The six questions were carefully graded, starting with a simple example that all should be able to do, moving on to a specific example that the experimental students would have met, then on to complex examples that would be unfamiliar to them all. These included a graph with a "corner", a vertical cusp and a function given by an unfamiliar formula. The last two questions concern functions given by different formulae on different parts of their domain, such as

$$y = \begin{cases} x & (x \leq 0) \\ x+x^2 & (x \geq 0) \end{cases}$$

Although GRADIENT has an input routine to cope with this kind of function, the experimental students were not given this information. They were under the impression that such a function could not be typed into the computer program and so it represented a genuine challenge.

The test was administered as a straight paper and pencil exercise for the control students whilst the experimental students were given the option of using the computer if they so desired. The Kenilworth experimental students felt that they did not need the computer, but those at Barton Peverill were encouraged to use it. In BE2 two students used it individually, one to zoom in on the graph y=sqr(abs(x)) at the origin, the other to check that the gradient of $y=x$ to the left of the origin was the same as that of $y=x+x^2$ to the right. In group BE2 the teacher was determined that the investigation would be performed using the computer in small groups, whilst individuals were urged to write down only their own interpretations. For this reason it may be useful initially to record the separate performances of each class.

The graph of $y=x^2-x$ at the origin

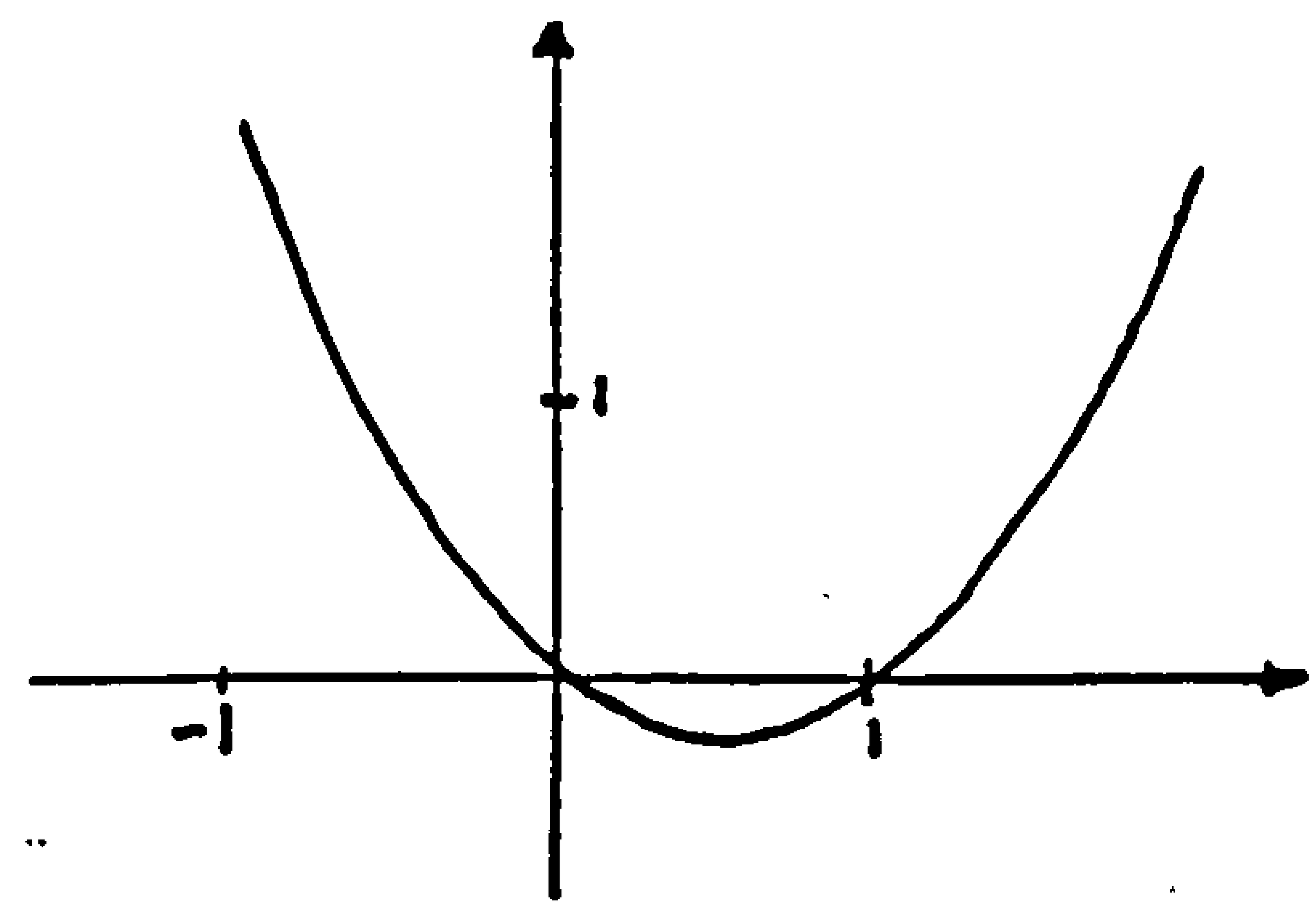



Figure 12.1

This was given as a simple starter to establish a little confidence amongst the students. The individual classes performed as in table 12.2 (with results in descending order of performance determined earlier in table 9.11). The number of students in each group may differ from those given previously because students absent from this or the tangent investigation, which follows, were eliminated from the figures. The university group U2 is a ~~totally~~ different subset from that mentioned in previous chapters, drawn from the same first year population.

| | | correct | incorrect | nr |
|---|---|---|---|---|
| BE2 | (N=16) | 16 | 0 | 0 |
| KE | (N=14) | 14 | 0 | 0 |
| BE1 | (N=11) | 10 | 1 | 0 |
| Total | (N=41) | 40 | 1 | 0 |
| | | | | |
| BC2 | (N=17) | 15 | 2 | 0 |
| KC | (N= 9) | 8 | 1 | 0 |
| BC4 | (N=11) | 11 | 0 | 0 |
| BC3 | (N=15) | 13 | 2 | 0 |
| BC1 | (N=13) | 12 | 1 | 0 |
| Total | (N=65) | 57 | 6 | 0 |
| | | | | |
| U2 | (N=47) | 47 | 0 | 0 |

Table 12.2

On this table there is a slight difference between the experimental and control groups. Performing a $X^2$-test with the null hypothesis that the control and experimental groups give the same distribution of correct responses, the breakdown, 40:1 experimental, 57:6 control, gives $X^2$=1.02, which is not significant.

The graph of y=abs(x) at the origin

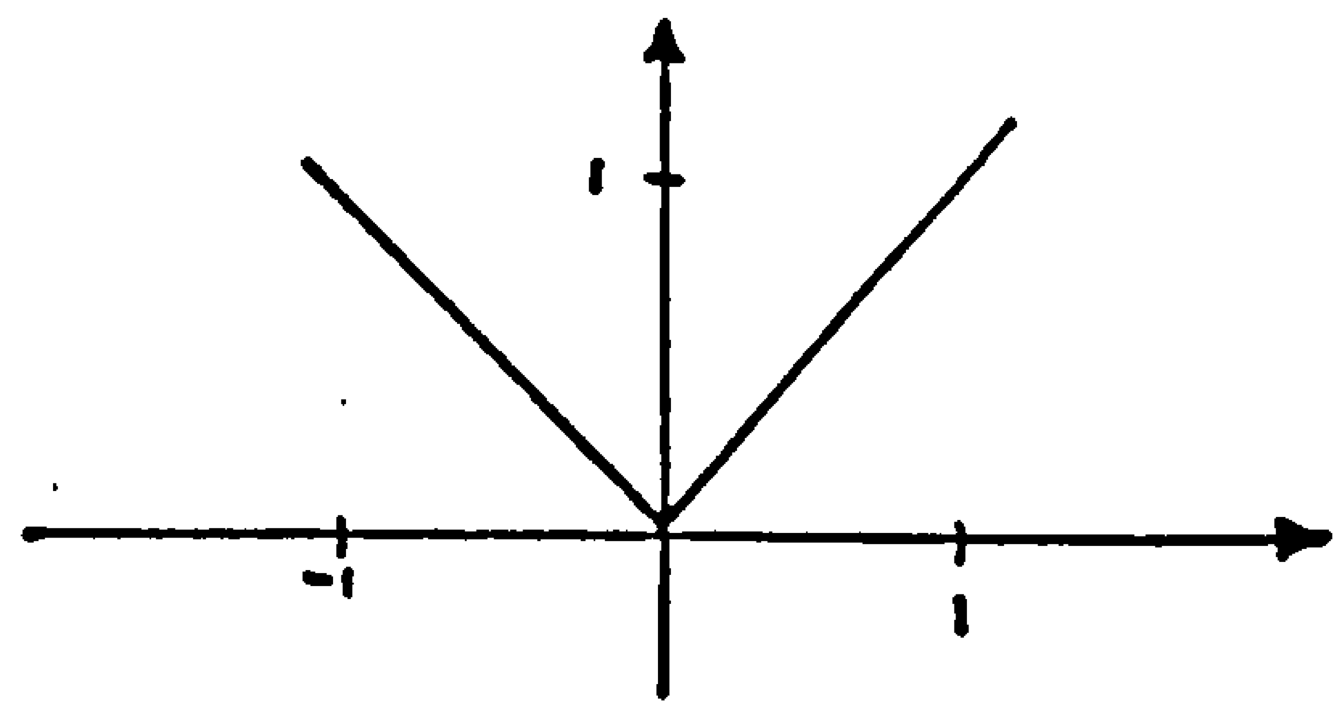



Figure 12.3

This graph has different left and right gradients at the origin and the experimental students will have considered it and learnt that at such a point the graph will be decreed not to have a gradient.

The majority of these students respond either in this manner, or note that the graph will not magnify to give a straight line:

> There are two different gradients when taken from the left and right of x=0 which will not correspond with each other (BE110*)

> It is not a straight line when magnified. Therefore gradient cannot be found. (BE104*)

Because $x=0$ is a corner on the graph. (KE12)

A small number hold to old concept images:

At $x=0$ many tangents are possible (KE09)

and three give a type of response that initially looks strange:

No, because the gradient is infinite. (BE214*)

Perhaps the following explanation:

Infinite rate of change of gradient, (U63)

from one of the university students gives a clue. The "infinite" idea may be related to the large *change* in gradient in any tiny interval enclosing the origin.

The control students were relying on their concept image of gradient and responded in several different ways. Many said the gradient could be calculated and gave the gradient as 0, 1 or ±1, with a few giving other responses (either a guess, or some explanation in terms of the formulae for the derivative for $x$ negative and $x$ positive). Those giving the value zero sometimes explicitly calculated the left and right gradient and averaged the result. This "average gradient" corresponds to what I shall call a "balance tangent", which sits evenly on the corner of the curve. Those finding the value 1 either look at the right gradient only, or attempt to differentiate the "abs" function. They may either ignore the symbol "abs" altogether and differentiate x to get the value 1, or differentiate abs(x) erroneously to obtain the value abs(1). Those giving ±1 either note that the absolute value is positive or negative, or they work out both gradients and consider the gradient to have two values.

The "NO" responses from the control students are given a variety of reasons. One is simply that they don't know how to cope with it:

> No - can't work out dy/dx of abs. (BC315*)

But others show that they have the insight to understand about the different left and right gradients:

> No, because the graph at x=0 is a right angle. (KC05)

> No, because the line is going two directions at 90 degrees. (BC202*)

> No, because at x=0 there's a point - therefore it has no gradient. (BC304*)

The latter initially confused me, until I saw the same kind of response repeated in other replies. It became clear that some of the students were referring to a "point" not in its mathematical sense, but in its colloquial sense as in a "point on a sword". This "point" is none other than a cusp, or corner...

The university students tend to give more sophisticated responses indicative of ideas they will have met later in the sixth form, some using continuity in a precise form:

> Gradient function is not continuous, undefined at x=0, (U63)

and others in an unorthodox manner mentioned in Tall & Vinner [1981]:

> Not continuous curve: no tangent. (U60)

In this latter form, a "continuous" curve may mean one that looks smooth, or that is given by a single formula.

The responses are classified in table 12.4:

| | | YES: 0 | 1 | ±1 | other | NO | nr |
|---|---|---|---|---|---|---|---|
| BE2 | (N=16) | 1 | 0 | 0 | 0 | 15 | 0 |
| KE | (N=14) | 1 | 0 | 0 | 1 | 12 | 0 |
| BE1 | (N=11) | 0 | 0 | 0 | 0 | 11 | 0 |
| Total | (N=41) | 2 | 0 | 0 | 1 | 38 | 0 |
| | | | | | | | |
| BC2 | (N=17) | 8 | 3 | 0 | 0 | 6 | 0 |
| KC | (N= 9) | 7 | 0 | 0 | 0 | 3 | 0 |
| BC4 | (N=11) | 3 | 2 | 1 | 0 | 5 | 0 |
| BC3 | (N=15) | 4 | 6 | 0 | 0 | 5 | 0 |
| BC1 | (N=13) | 1 | 3 | 3 | 3 | 3 | 0 |
| Total | (N=65) | 23 | 14 | 4 | 3 | 21 | 0 |
| | | | | | | | |
| U2 | (N=47) | 7 | 1 | 2 | 1 | 36 | 0 |

Table 12.4

Clearly there is little difference between the three experimental groups, all of whom have only one "YES" response. By contrast, all the control groups have a *majority* of "YES" responses.

Contrasting the "NO" category with those who responded otherwise gives table 12.5:

| | NO | other |
|---|---|---|
| Experimental | 38 | 3 |
| Control | 21 | 44 |
| University | 36 | 11 |

Table 12.5

Using a $\chi^2$-test, with the null hypothesis that the distributions of the experimental and control groups are the same, gives $\chi^2=34.73$, which is massively significant (better than 0.0001%). The same comparison between experimental students and those at university gives $\chi^2=3.12$, which is not significant at the 5% level (though it would have been had we specified a one-tail test).

The graph of y=√(abs(x)) at the origin

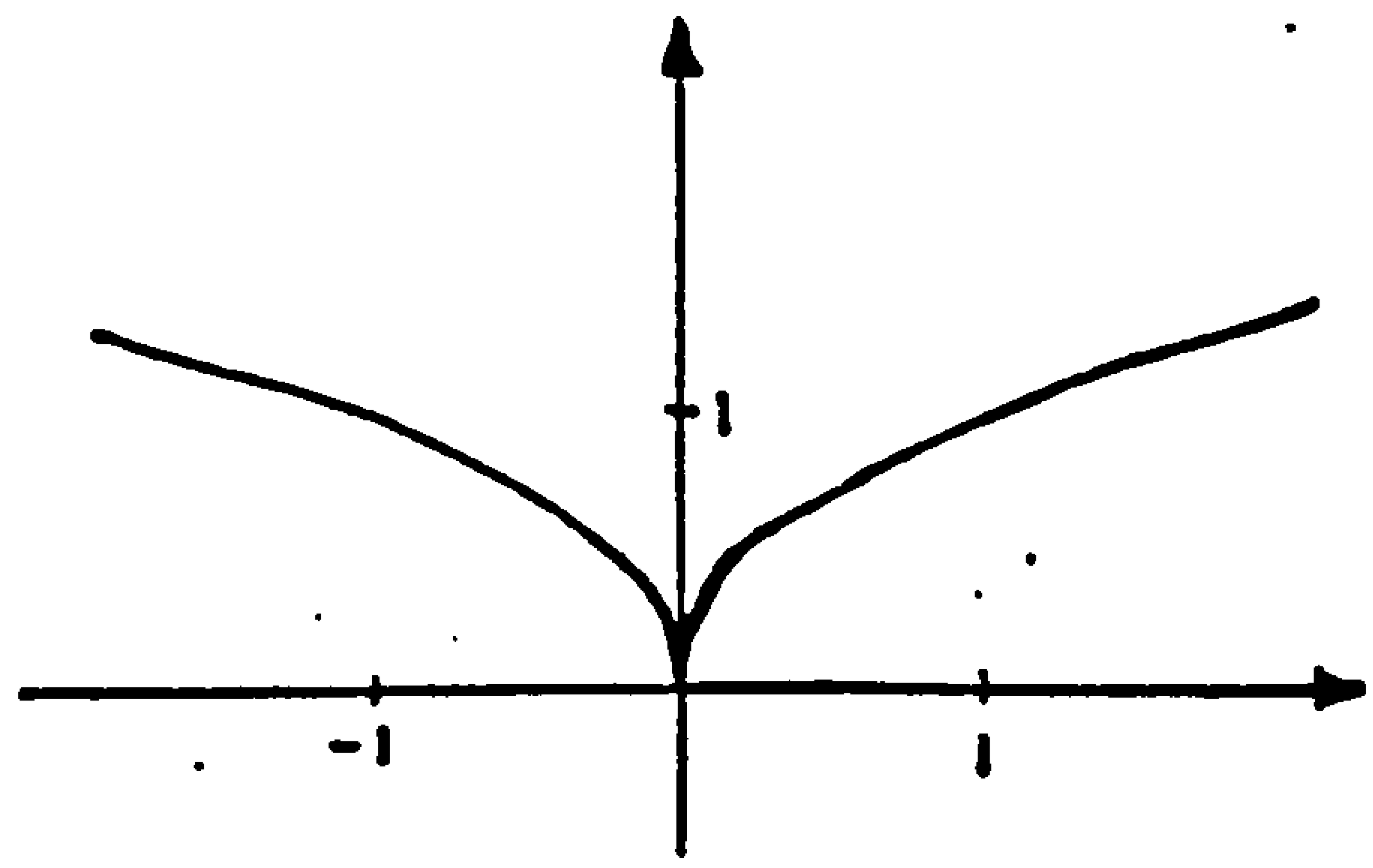



Figure 12.6

This graph has a vertical cusp at the origin. If the gradient is calculated from the left, the limit is $-\infty$ and from the right it is $+\infty$. Thus one may claim that the gradient is "infinite" or one may say that $+\infty$ and $-\infty$ are different, so there is no unique gradient. Alternatively, one may claim that the limit does not exist because infinity is not a real number. Another viewpoint, shared by several control students, involves calculating the derivative (perhaps forgetting the "abs") to get $\frac{1}{2}x^{-1/2}$, putting $x=0$ and claiming that 1/0 "does not exist".

If one visualizes a point moving along the curve, it does not move smoothly at the origin, but reverses direction. This suggests that if one regards the gradient as a rate of change, then the gradient is not properly defined at the origin. Likewise, if one magnifies the graph at the origin, it never magnifies to a point. My own preferred response to this question is that the gradient is not defined, but there are some mathematicians who may well quarrel with this interpretation.

The experimental students respond in much the same vein as the previous case, either by noting different gradients to left and right, which they may describe as $-\infty$ and $+\infty$, or by noting that it will not magnify to give a straight line. The control students again give a wider spectrum of replies.

In table 12.7 the responses that the gradient can be calculated are subdivided into three: the first say that the gradient is infinite, the second that it is zero, without any working, and the third give other responses, usually erroneous attempts to calculate the derivative. Some of the latter calculate the derivative formula correctly as $\frac{1}{2}x^{-1/2}$ and then put $x=0$ to get the erroneous answer zero. These are not classified with those who say it is zero directly as some of the latter may be visualizing the gradient as that of a "balance tangent", or "average tangent", sitting on the base of the cusp.

Those replying "NO" are again sub-divided, into those that give this response because an infinite limit cannot be calculated and those who give another reason.

| | | YES: ∞ | 0 | other | NO: ∞ | other | nr |
|---|---|---|---|---|---|---|---|
| BE2 | (N=16) | 5 | 0 | 0 | 3 | 8 | 0 |
| KE | (N=14) | 3 | 1 | 0 | 2 | 8 | 0 |
| BE1 | (N=11) | 0 | 0 | 0 | 0 | 11 | 0 |
| Total | (N=41) | 8 | 1 | 0 | 5 | 27 | 0 |
| | | | | | | | |
| BC2 | (N=17) | 2 | 1 | 2 | 1 | 10 | 1 |
| KC | (N= 9) | 3 | 5 | 0 | 0 | 1 | 0 |
| BC4 | (N=11) | 2 | 3 | 0 | 2 | 4 | 0 |
| BC3 | (N=15) | 1 | 1 | 6 | 3 | 3 | 1 |
| BC1 | (N=13) | 2 | 1 | 8 | 0 | 2 | 0 |
| Total | (N=65) | 10 | 11 | 16 | 6 | 20 | 2 |
| | | | | | | | |
| U2 | (N=47) | 14 | 5 | 4 | 6 | 18 | 0 |

Table 12.7

Here there *is* a significant difference between the three experimental groups. BE1 has all its responses in the "NO" category (with none responding "NO:∞"). BE2 and KE, on the other hand, split almost equally between the "NO:other" category and "YES:∞" or "NO:∞". The difference between the groups is because the BE1 students look upon the graph at the origin as a "corner" or as a graph which "will not magnify to give a straight line". The BE2 and KE students, on the other hand, may visualize the extended chord through the origin A and a nearby point B tending to the vertical position as B→A, but there is a difference of opinion as to whether the value ∞ is acceptable or not.

There are two useful ways of grouping these results together: the first is given in table 12.8, contrasting those who say "NO" (without mentioning infinity) compared with all other responses.

| | NO | other |
|---|---|---|
| Experimental | 27 | 14 |
| Control | 20 | 45 |
| University | 18 | 29 |

Table 12.8

A $\chi^2$-test comparing the distributions of experimental and control, with the null hypothesis that the distributions are the same, gives $\chi^2=11.16$, which is significant at the 0.1% level. The same comparison between experimental students and those at university gives $\chi^2=5.16$, which is significant at the 5% level.

A second sub-division might be to group those who say "NO" or who give the response "∞" (with a yes or no). This is given in table 12.9:

| | NO/∞ | other |
|---|---|---|
| Experimental | 40 | 1 |
| Control | 35 | 29 |
| University | 38 | 9 |

Table 12.9

A $\chi^2$-test with the null hypothesis that the distributions of the experimental and control groups are the same gives $\chi^2=20.46$, significant at the 0.01% level. The same test comparing experimental and university students gives $\chi^2=4.52$, significant at the 5% level.

Thus the experimental students give a more coherent response to this question (in the terms described earlier) than both the control and university students.

The graph of y=abs($x^3$) at the origin

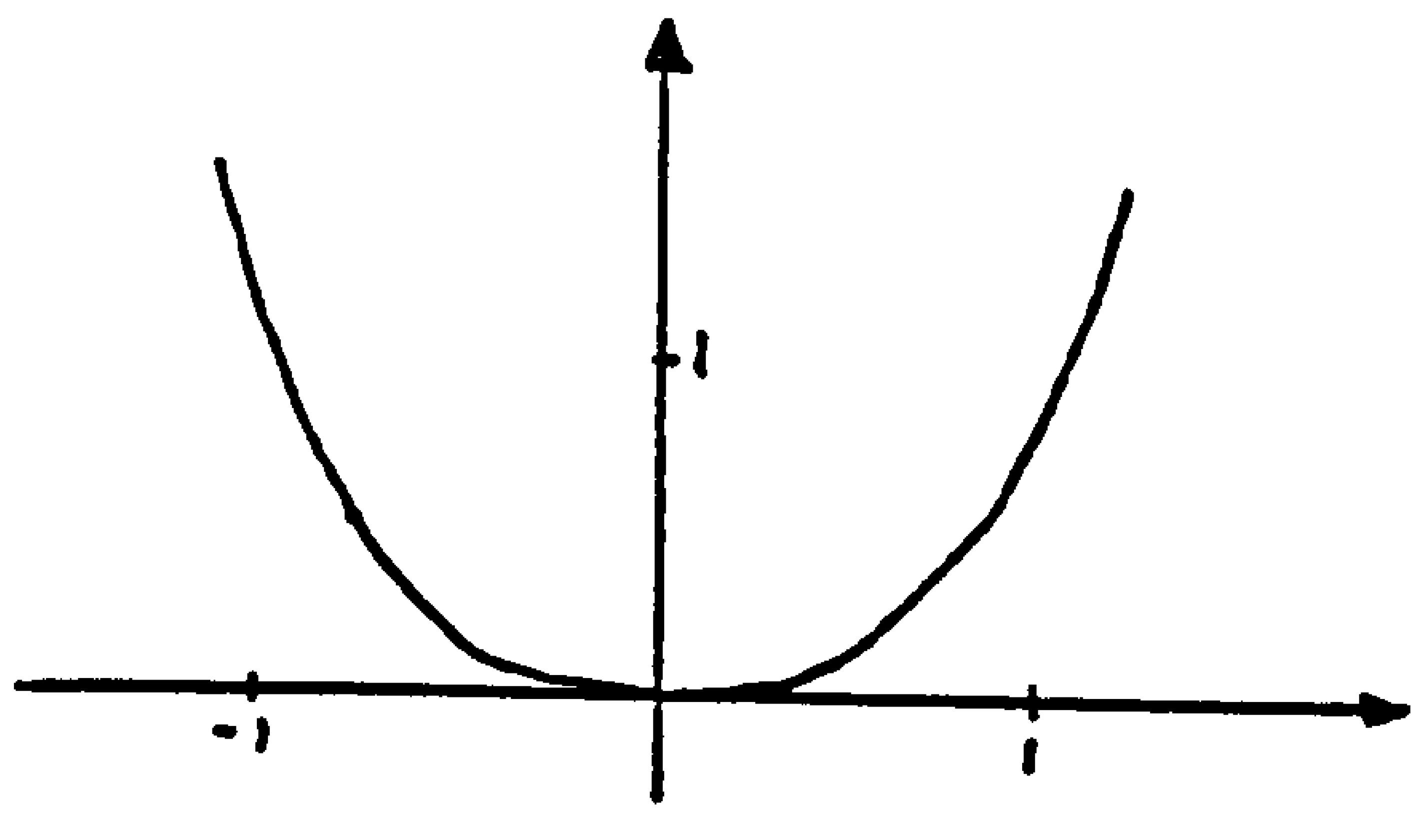



Figure 12.10

The problem here is the strangeness of the function abs($x^3$), for the picture clearly has gradient zero at the origin. The experimental students may be able to see what is happening from the picture, or could type the function into the computer to obtain a numerical answer. The control students will be more likely to attack the problem by differentiating the formula. This is fraught with difficulty, for most either ignore the "abs" and get the derivative to be $3x^2$, or they differentiate inside the "abs" function to get the "derivative" abs($3x^2$) which has the wrong sign for $x<0$. (A correct formula might be 3x(abs(x))...)

The responses are given in table 12.11, with those giving the gradient as zero being subdivided into those simply writing down the answer (or using the correct formula) and those getting the answer zero from an incorrect formula (denoted by 0(?)). This division may be somewhat arbitrary in that some responses included in the first category may implicitly be using a wrong formula. Nevertheless, the table of responses with this subdivision makes interesting reading.

| | | YES: 0 | 0(?) | other | NO | nr |
|---|---|---|---|---|---|---|
| BE2 | (N=16) | 14 | 1 | 0 | 1 | 0 |
| KE | (N=14) | 13 | 1 | 0 | 0 | 0 |
| BE1 | (N=11) | 8 | 3 | 0 | 0 | 0 |
| Total | (N=41) | 35 | 5 | 0 | 1 | 0 |
| | | | | | | |
| BC2 | (N=17) | 13 | 2 | 0 | 2 | 0 |
| KC | (N= 9) | 9 | 0 | 0 | 0 | 0 |
| BC4 | (N=11) | 5 | 6 | 0 | 0 | 0 |
| BC3 | (N=15) | 3 | 12 | 0 | 0 | 0 |
| BC1 | (N=13) | 6 | 2 | 5 | 0 | 0 |
| Total | (N=65) | 36 | 22 | 5 | 2 | 0 |
| | | | | | | |
| U2 | (N=47) | 35 | 9 | 1 | 2 | 0 |

Table 12.11

Clearly the experimental students give far more responses in the first column. Grouping the responses together into first column versus the rest gives table 12.12:

| | gradient=0 | other |
|---|---|---|
| Experimental | 35 | 6 |
| Control | 36 | 29 |
| University | 35 | 12 |

Table 12.12

A $\chi^2$-test with the null hypothesis that the distributions of the experimental and control groups are the same gives $X^2=8.90$, significant at the 1% level. The same test comparing experimental and university students gives $X^2=1.00$, which is *not* significant. Thus the experimental students are again performing at a better level than the control students, and at a level comparable with those at university.

The graph of $y=\begin{cases} x & (x\leq 0) \\ x+x^2 & (x\geq 0) \end{cases}$ at the origin

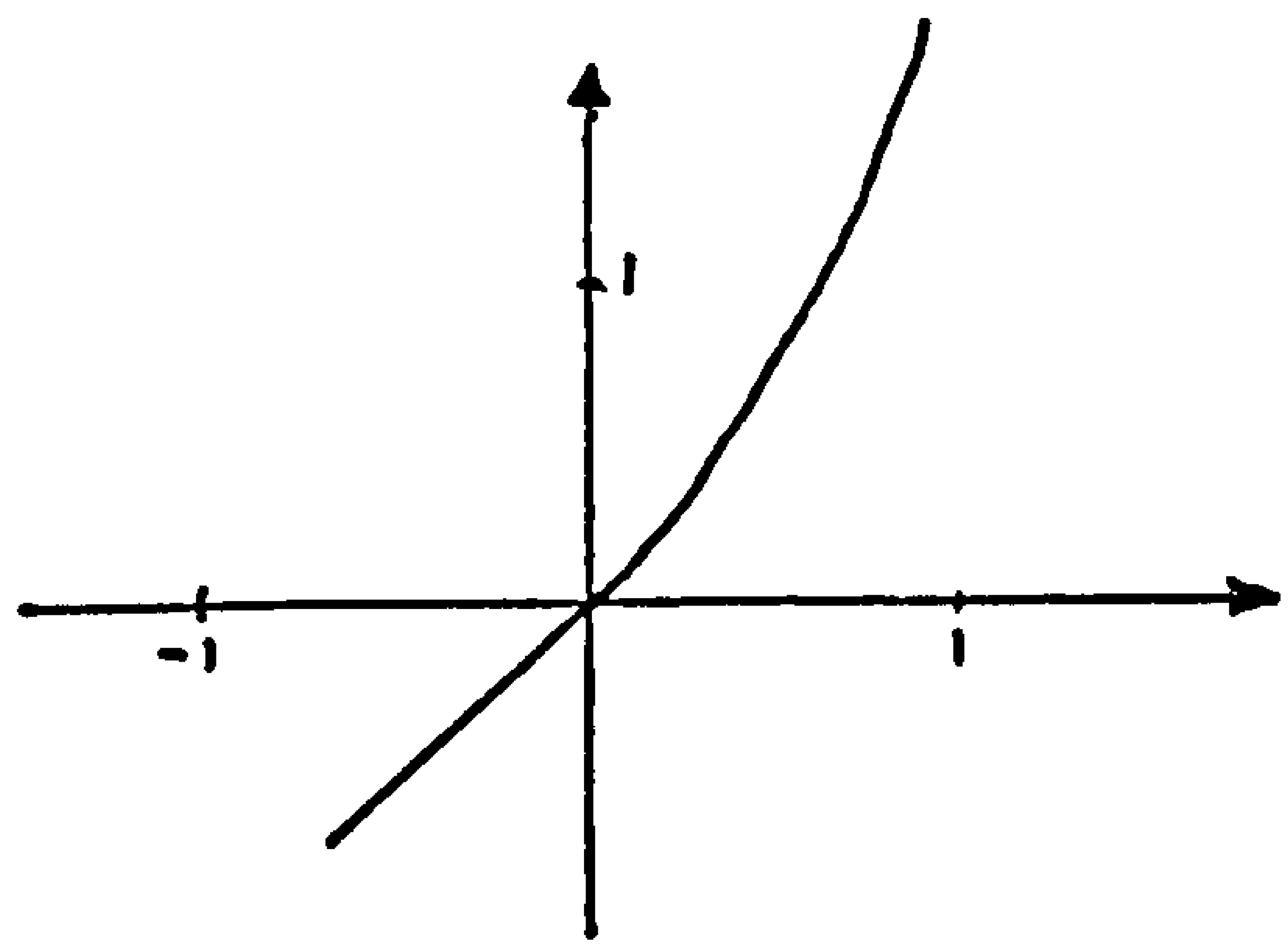



Figure 12.13

This question causes some problems because there are different formulae on either side of the origin. Those using the computer do not realise that they can type in the formula in this form and so can only work out the gradients numerically by drawing separate graphs for $y=x$ and, later, for $y=x+x^2$. This is now a test of their understanding of the underlying idea of gradient. The control students, on the other hand, may be more likely to attempt to differentiate the formulae, and consider either that they do not know how to calculate the gradient, or that the gradient cannot be calculated:

> Because at x=0 is where two functions meet (BC205*)

or because

> The line changes its characteristics - it is two graphs (BC211*).

The responses are given in table 12.14, the "YES" responses being subdivided into those giving the correct answer 1 (either written straight down or calculated using differentiation of both formulae), and those responding otherwise.

| | | YES: 1 | other | NO | nr |
|---|---|---|---|---|---|
| BE2 | (N=16) | 16 | 0 | 0 | 0 |
| KE | (N=14) | 13 | 1 | 0 | 0 |
| BE1 | (N=11) | 10 | 0 | 1 | 0 |
| Total | (N=41) | 39 | 1 | 1 | 0 |
| | | | | | |
| BC2 | (N=17) | 6 | 2 | 9 | 0 |
| KC | (N= 9) | 7 | 1 | 1 | 0 |
| BC4 | (N=11) | 5 | 0 | 6 | 0 |
| BC3 | (N=15) | 6 | 1 | 7 | 1 |
| BC1 | (N=13) | 8 | 4 | 1 | 0 |
| Total | (N=65) | 32 | 8 | 24 | 1 |
| | | | | | |
| U2 | (N=47) | 45 | 0 | 2 | 0 |

Table 12.14

Again the experimental students give proportionately more responses in the first column. Comparing the numbers of students giving the correct "YES" response against all the rest of the responses gives table 12.15:

| | YES:1 | other |
|---|---|---|
| Experimental | 39 | 2 |
| Control | 32 | 33 |
| University | 45 | 2 |

Table 12.15

A $\chi^2$-test with the null hypothesis that the distributions of the experimental and control groups are the same gives $\chi^2$=21.91, significant at the 0.01% level. There is visibly little difference between the experimental and university students ($\chi^2$=0.14). Again the experimental students can cope with this type of question significantly better than the control students, and at a level comparable with those at university.

The graph of $y = \begin{cases} x & (x \le 0) \\ x^2 & (x \ge 0) \end{cases}$ at the origin

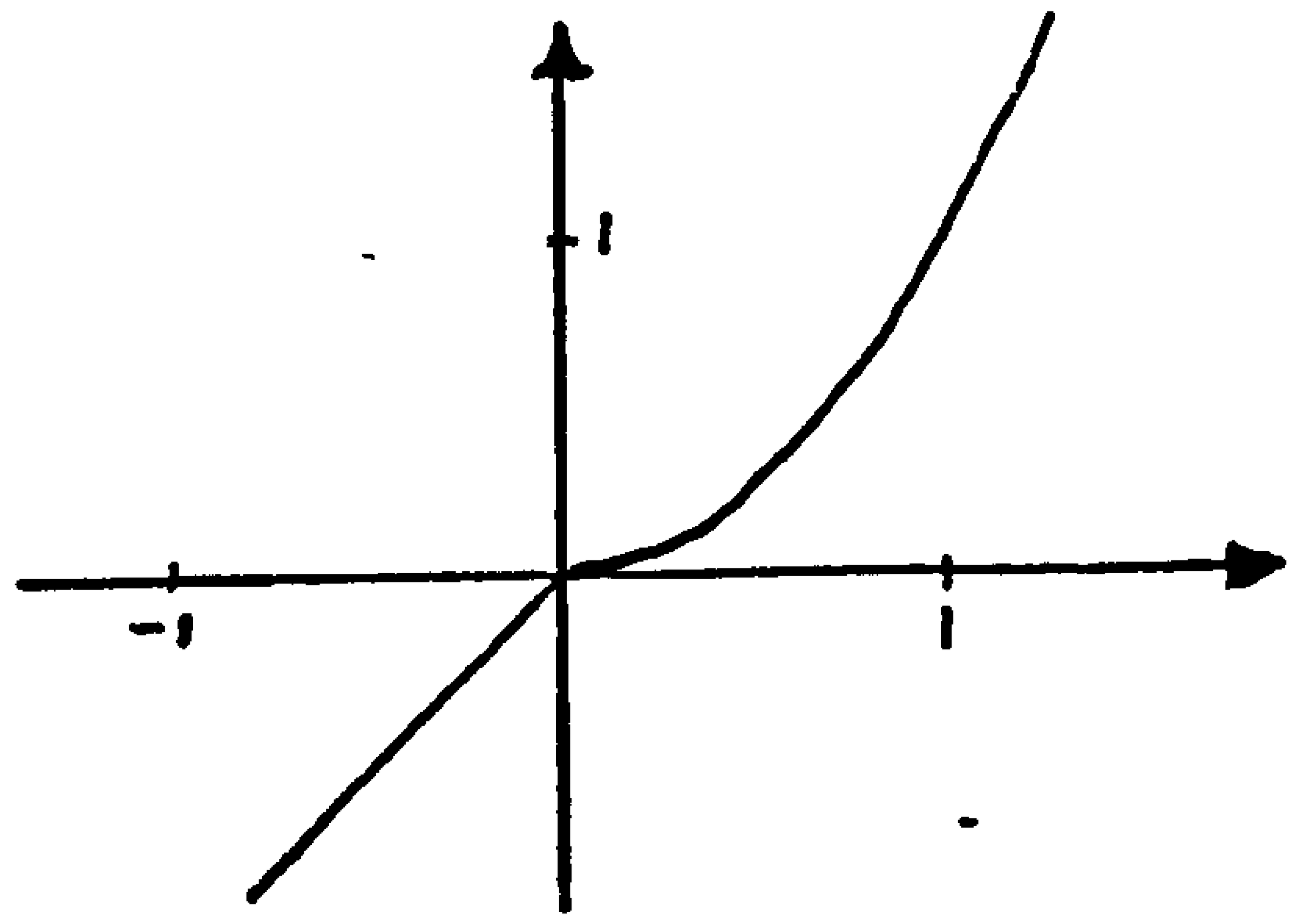



Figure 12.16

This final example does not have a gradient defined at the origin because the left and right derivatives are different. The experimental students in the main correctly respond that there is no (single) value of the gradient, usually giving one of the following reasons: the different left and right gradients, or the fact that the graph has a corner, or that it does not magnify to a straight line.

The control students give a wider variety of responses. Some say "YES", the gradient can be calculated, either by considering it to be two-valued, or by taking the value on one side only, or by calculating left and right gradients and averaging.

The control students who answer "NO", may either do so because they know that the gradient is not defined at the origin or, alternatively, because they do not know how to respond to the question. Fortunately the statistics are so clear that this difference does not cause difficulties of interpretation (table 12.17).

| | | NO | YES | nr |
|---|---|---|---|---|
| BE2 | (N=16) | 15 | 1 | 0 |
| KE | (N=14) | 10 | 4 | 0 |
| BE1 | (N=11) | 11 | 0 | 0 |
| Total | (N=41) | 36 | 5 | 0 |
| | | | | |
| BC2 | (N=17) | 15 | 2 | 0 |
| KC | (N= 9) | 6 | 3 | 0 |
| BC4 | (N=11) | 10 | 1 | 0 |
| BC3 | (N=15) | 4 | 9 | 2 |
| BC1 | (N=13) | 4 | 9 | 0 |
| Total | (N=65) | 39 | 24 | 2 |
| | | | | |
| U2 | (N=47) | 43 | 3 | 1 |

Table 12.17

There are (proportionately) more experimental students responding "NO"; grouping together "YES" and "nr" gives table 12.18:

| | NO | other |
|---|---|---|
| Experimental | 36 | 5 |
| Control | 39 | 26 |
| University | 43 | 4 |

Table 12.18

A $\chi^2$-test with the null hypothesis that the distributions of the experimental and control groups are the same gives $\chi^2=8.09$, significant at the 1% level, whilst there is clearly no significant difference between the experimental students and those at university ($\chi^2=0.47$).

## The Tangent Investigation

This follows the same pattern as the gradient investigation and it was planned to be given a week or so later. It had the same six functions and the same pictures, the only difference being the format of the question which said:

> Does the graph have a tangent at $x=0$? YES/NO
> If YES, please sketch the tangent? If NO, why not?

Would this small change produce very different answers?

### The graph of $y=x^2-x$ at the origin

The starter question produced a higher rate of success than the first questionnaire and there is clearly no significant difference between the groups (table 12.19).

| | | correct | incorrect | nr |
|---|---|---|---|---|
| BE2 | (N=16) | 16 | 0 | 0 |
| KE | (N=14) | 14 | 0 | 0 |
| BE1 | (N=11) | 11 | 0 | 0 |
| Total | (N=41) | 41 | 0 | 0 |
| | | | | |
| BC2 | (N=17) | 17 | 0 | 0 |
| KC | (N= 9) | 9 | 0 | 0 |
| BC4 | (N=11) | 11 | 0 | 0 |
| BC3 | (N=15) | 15 | 0 | 0 |
| BC1 | (N=13) | 12 | 1 | 0 |
| Total | (N=65) | 64 | 1 | 0 |
| | | | | |
| U2 | (N=47) | 47 | 0 | 0 |

Table 12.19

The graph of y=abs(x) at the origin

A small number of students responded that this graph had "many" or "an infinite number" of tangents at the origin. There may have been more but for the slightly inopportune wording which asked if the graph had a tangent (implying only one). Some said there were "two", and this created a difficulty in interpretation. In the experimental groups we had talked about "left" and "right" tangents, and students responded in the following vein:

> You cannot have a tangent at the point. The tangent on the left is different from the tangent at the right. (KE02)

This clearly means there is *no* tangent, but others responded slightly more ambiguously:

> No, because there is a tangent either side of the point

which are different. (KE05)

Comparing this with responses from control students such as:

No. There are two tangents to the two lines (BC406)

it becomes difficult on the basis of the written word alone to distinguish between those who use this argument to specify there are "no" tangents and those who think there genuinely are "two". To avoid accusations of bias, all these cases will be treated as saying "there are two tangents". Fortunately there are so few amongst the experimental students that the differences between the groups remain statistically significant.

Students asserting there was a single tangent might sketch either the left, right, or the "balance tangent". In practice there were no left tangents: students in the experimental groups did not make this interpretation and students in the control groups usually experience positive increments in their calculus pictures. The "balance tangent", in this case the horizontal axis, was the most popular positive response.

The full classification is given in table 12.20:

| | | many | two | left | right | balance | none | nr |
|---|---|---|---|---|---|---|---|---|
| BE2 | (N=16) | 0 | 1 | 0 | 0 | 0 | 15 | 0 |
| KE | (N=14) | 2 | 4 | 0 | 0 | 1 | 7 | 0 |
| BE1 | (N=11) | 0 | 1 | 0 | 0 | 0 | 10 | 0 |
| Total | (N=41) | 2 | 5 | 0 | 0 | 1 | 32 | 0 |
| | | | | | | | | |
| BC2 | (N=17) | 6 | 4 | 0 | 0 | 0 | 7 | 0 |
| KC | (N= 9) | 0 | 0 | 0 | 0 | 5 | 4 | 0 |
| BC4 | (N=11) | 1 | 4 | 0 | 0 | 1 | 5 | 0 |
| BC3 | (N=15) | 0 | 1 | 0 | 2 | 5 | 7 | 0 |
| BC1 | (N=13) | 1 | 0 | 0 | 0 | 6 | 6 | 0 |
| Total | (N=65) | 8 | 9 | 0 | 2 | 17 | 29 | 0 |
| | | | | | | | | |
| U2 | (N=47) | 2 | 0 | 0 | 0 | 8 | 37 | 0 |

Table 12.20

Notice that there is a difference between the experimental group KE and the other two BE1, BE2, in that the breakdown of those saying "no tangent" is 7 out of 14 for KE and 10 out of 11, 15 out of 16 for BE1, BE2 respectively.

We shall return to this later...

Comparing the total numbers in experimental and control who responded that there were none and those who responded otherwise gives table 12.21:

| | none | other |
|---|---|---|
| Experimental | 32 | 9 |
| Control | 29 | 35 |
| University | 37 | 10 |

Table 12.21

Using a $X^2$-test, with the null hypothesis that the distributions of the experimental and control groups are the same, gives $X^2=9.70$, which is significant at the 1% level. There is visibly no significant difference between the performance of the experimental students and those at university ($\chi^2=0.03$).

The graph of y=√(abs(x)) at the origin

This graph gives an intriguing problem when the tangent is considered at the origin. When the extended chord AB is drawn through the origin A and a nearby point B on the curve, as B→A from either side the chord tends to the vertical position. Does the graph have a tangent? Some mathematicians will say "yes" and others "no" and there is room for genuine dispute.

A sophisticated response will be to look at a point moving along the curve. What is its direction at the origin? It turns through 180 degrees as it passes through the origin. The left gradient is $-\infty$ and the right gradient is $+\infty$. The slick answer is to say that the graph has an "undirected tangent" but not a "directed tangent". Support for this is given by magnifying at the origin. It does not magnify to a straight line, therefore it is structurally different from a regular curve with a tangent. If a student is using the magnification property, (s)he will say that the curve has no tangent there.

The responses from the students follow a similar spectrum to the previous question: some say there are curves with many tangents, some specify two (either because their gradients are $\pm\infty$ or because the student sees the graph as having two separate steep, but not vertical, tangents). A few students suggest "balance tangents" (though not as many as the previous graph) and a small number of others draw a line touching the inside of the cusp at a small angle to the right of the vertical.

The categories are as in table 12.22:

| | | many | two | vertical | balance | other | none | nr |
|---|---|---|---|---|---|---|---|---|
| BE2 | (N=16) | 0 | 0 | 0 | 1 | 0 | 15 | 0 |
| KE | (N=14) | 1 | 0 | 3 | 0 | 0 | 10 | 0 |
| BE1 | (N=11) | 0 | 0 | 0 | 0 | 0 | 11 | 0 |
| Total | (N=41) | 1 | 0 | 3 | 1 | 0 | 36 | 0 |
| BC2 | (N=17) | 2 | 0 | 5 | 1 | 1 | 8 | 0 |
| KC | (N= 9) | 0 | 0 | 5 | 3 | 0 | 1 | 0 |
| BC4 | (N=11) | 1 | 1 | 3 | 0 | 1 | 5 | 0 |
| BC3 | (N=15) | 0 | 0 | 4 | 4 | 1 | 5 | 1 |
| BC1 | (N=13) | 0 | 0 | 6 | 2 | 0 | 5 | 0 |
| Total | (N=65) | 3 | 1 | 23 | 10 | 3 | 24 | 1 |
| U2 | (N=47) | 2 | 1 | 23 | 2 | 0 | 19 | 0 |

Table 12.22

Grouping together all the categories other than "none" gives table 12.23:

| | none | other |
|---|---|---|
| Experimental | 36 | 5 |
| Control | 24 | 41 |
| University | 19 | 28 |

Table 12.23

A $\chi^2$-test comparing the experimental and control categories, with the null hypothesis that the distributions are the same, gives $\chi^2=17.71$, which is significant at the 0.01% level. The same comparison between experimental students and those at university gives $X^2=7.43$, which is also significant at the 1% level. Thus the experimental students give a more coherent response to this question (in the terms described earlier) than both the control and university students.

As both "none" and "vertical" could be appropriate mathematical responses, we will re-group these together and compare with the remainder (table 12.24):

| | none/vertical | other |
|---|---|---|
| Experimental | 39 | 2 |
| Control | 47 | 18 |
| University | 42 | 5 |

Table 12.24

Using the same $X^2$-test for the experimental and control groups gives $X^2=7.12$, significant at the 0.1% level, whilst the difference between the experimental and university students is clearly insignificant ($X^2=0.36$).

The graph of $y=abs(x^3)$ at the origin

In this case, with only the tangent to sketch, most of the students drew the correct horizontal tangent (table 12.25).

| | | horizontal | other | none | nr |
|---|---|---|---|---|---|
| BE2 | (N=16) | 16 | 1 | 0 | 0 |
| KE | (N=14) | 13 | 1 | 0 | 0 |
| BE1 | (N=11) | 11 | 0 | 0 | 0 |
| Total | (N=41) | 39 | 2 | 0 | 0 |
| | | | | | |
| BC2 | (N=17) | 17 | 0 | 0 | 0 |
| KC | (N= 9) | 9 | 0 | 0 | 0 |
| BC4 | (N=11) | 11 | 0 | 0 | 0 |
| BC3 | (N=15) | 13 | 1 | 1 | 0 |
| BC1 | (N=13) | 12 | 0 | 1 | 0 |
| Total | (N=65) | 62 | 1 | 2 | 0 |
| | | | | | |
| U2 | (N=47) | 46 | 0 | 1 | 0 |

Table 12.25

There is clearly no significant difference.

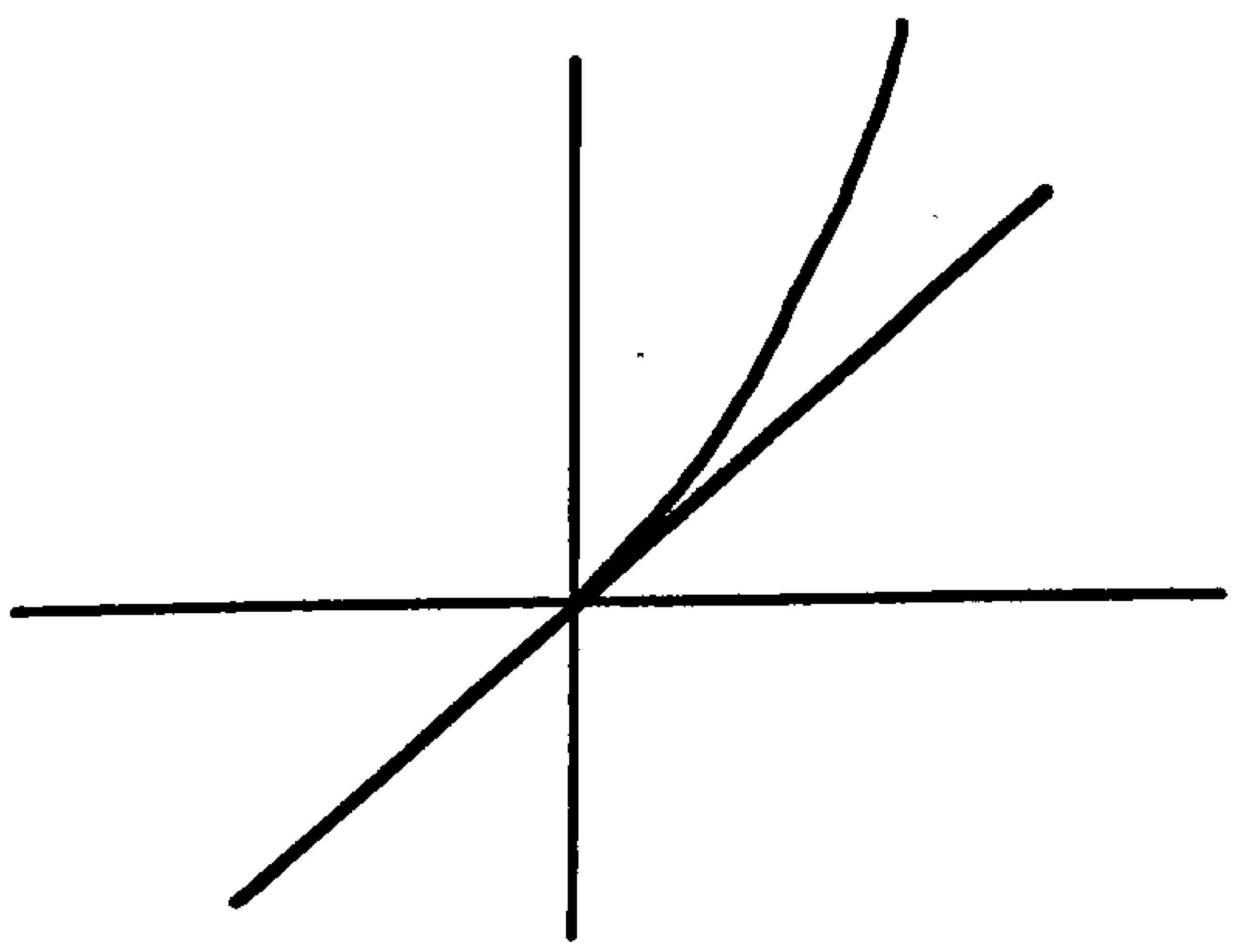
The graph of $y= \begin{cases} x & (x\le 0) \\ x+x^2 & (x\ge 0) \end{cases}$ at the origin

In this problem the question of the naive definition of a tangent raises its ugly head. In this definition a tangent is a line which touches the curve at one point only and does not cross. Students who interpret a tangent in terms of this concept image may not be able to draw the official tangent, because it is x=y and coincides with the part of the graph to the left of the origin.

One way out is to insist that there is no tangent:

Because the tangent should touch the line at one specific point but this tangent would touch it constantly. (BC304)

Alternatively the problem is avoided by drawing the tangent not as in Figure 12.26(a), but as in 12.26(b).


(a)

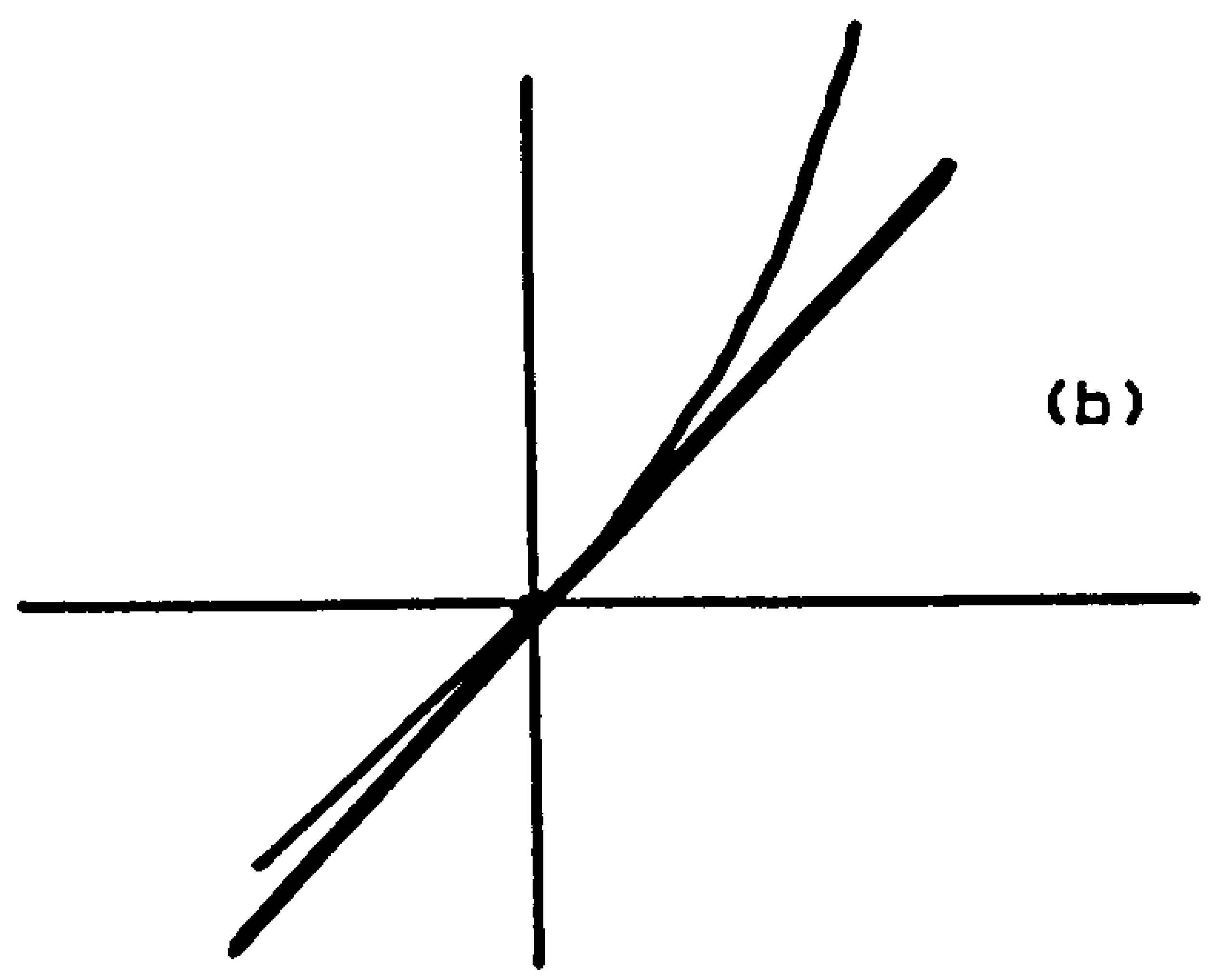

(b)

Figure 12.26

This naive notion of tangent is essentially another example of a *generic* concept, discussed earlier in chapter 2. It is the accretion of all the "typical" examples of tangents that "cut at one point only" but "do not cross".

There is also an additional complication. The function is given by two different formulae and some students are not able to cope with this new situation:

The graph is two separate functions and there is not a tangent at x=0. (BC213*)

Thus we see three main possible categories: the standard tangent, the naive generic tangent, or no tangent at all. In addition there is an "other" category, to cope with two students who insist there *is* a tangent, but fail to draw it (table 12.27).

| | | standard | generic | other | none | nr |
|---|---|---|---|---|---|---|
| BE2 | (N=16) | 11 | 5 | 0 | 0 | 0 |
| KE | (N=14) | 11 | 2 | 0 | 1 | 0 |
| BE1 | (N=11) | 9 | 1 | 0 | 1 | 0 |
| Total | (N=41) | 31 | 8 | 0 | 2 | 0 |
| BC2 | (N=17) | 3 | 9 | 0 | 5 | 0 |
| KC | (N= 9) | 1 | 8 | 0 | 0 | 0 |
| BC4 | (N=11) | 4 | 1 | 2 | 4 | 0 |
| BC3 | (N=15) | 7 | 5 | 0 | 2 | 1 |
| BC1 | (N=13) | 7 | 2 | 0 | 4 | 0 |
| Total | (N=65) | 22 | 30 | 2 | 15 | 1 |
| U2 | (N=47) | 29 | 14 | 0 | 4 | 0 |

Table 12.27

There are many generic tangents, showing the coercive effects of the concept image of the naive definition of a tangent. There are 8 out of 41 experimental students giving a generic response (20%) which, though substantial, is considerably smaller than the 46% of control students (30 out of 65) with the same conception. The generic limit persists through to university in 14 out of 47 students (30% of the sample).

Contrasting the correct (standard) responses with all others gives table 12.28:

| | standard | other |
|---|---|---|
| Experimental | 31 | 10 |
| Control | 22 | 43 |
| University | 29 | 18 |

Table 12.28

A $X^2$-test with the null hypothesis that the distributions of the experimental and control groups are the same gives $X^2=15.91$, significant at the 0.01% level. There is no significant difference between the experimental and university students ($X^2=1.36$).

The graph of $y=\begin{cases} x & (x\leq 0) \\ x^2 & (x\geq 0)\end{cases}$ at the origin

In the final example, the control students again mainly give the standard response that there is no tangent at the origin (with the same technical problem of interpretation that some bolster their argument by saying that there are two different tangents to the left and right). The experimental students on the other hand, leaven the standard description with a number of individuals giving an array of left, right and "balance" tangents. In this case there are two distinct pictures for a balance tangent. One neatly balances at a rakish angle on the curve at the origin, and another attempt passes through the origin and settles just below the right hand curve. (Figure

12.29.)

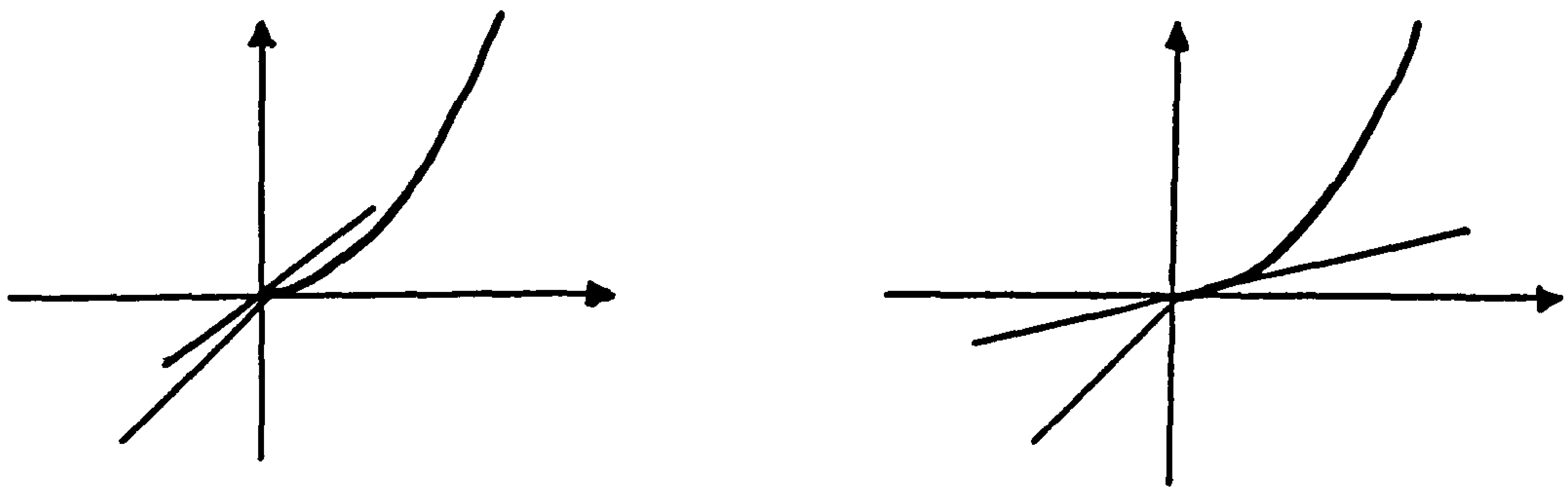


Figure 12.29

The full statistics are given in table 12.30:

| | | many | two | left | right | balance | other | none | nr |
|---|---|---|---|---|---|---|---|---|---|
| BE2 | (N=16) | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 0 |
| KE | (N=14) | 0 | 3 | 0 | 0 | 0 | 0 | 11 | 0 |
| BE1 | (N=11) | 0 | 0 | 0 | 0 | 1 | 0 | 10 | 0 |
| Total | (N=41) | 0 | 3 | 0 | 0 | 1 | 0 | 37 | 0 |
| BC2 | (N=17) | 2 | 1 | 0 | 1 | 1 | 0 | 12 | 0 |
| KC | (N= 9) | 0 | 0 | 0 | 1 | 3 | 0 | 5 | 0 |
| BC4 | (N=11) | 1 | 2 | 1 | 0 | 0 | 0 | 7 | 0 |
| BC3 | (N=15) | 0 | 1 | 1 | 1 | 2 | 2 | 5 | 3 |
| BC1 | (N=13) | 0 | 0 | 0 | 1 | 1 | 2 | 9 | 0 |
| Total | (N=65) | 3 | 4 | 2 | 4 | 7 | 4 | 38 | 3 |
| U2 | (N=47) | 1 | 4 | 0 | 1 | 2 | 0 | 38 | 1 |

Table 12.30

Proportionately more experimental students respond "none" than control students. Grouping together the remaining categories gives table 12.31:

| | NO | other |
|---|---|---|
| Experimental | 37 | 4 |
| Control | 38 | 27 |
| University | 38 | 9 |

Table 12.31

Testing the null hypothesis that the distribution of experimental and control students are the same gives $X^2=10.79$, significant at the 1% level, whilst there is clearly no significant difference between the experimental students and those at university ($X^2=0.88$).

## Comparison of groups of matched pairs

It is important to check whether the differences between experimental and control noted in the previous two sections hold good for both those with and without previous calculus experience. If there is no difference, then one may be more secure in the belief that the method is viable both for students studying calculus for the first time, as well as for those returning to it after earlier study. We therefore briefly carry out analogous calculations to those in the previous two sections for the matched pairs selected in chapter nine.

Four students used earlier in the matching missed one or other of the investigations. These were eliminated, together with the corresponding matched students, leaving eleven pairs without previous calculus experience and twenty four with experience. A

$X^2$-test requires a drastic difference in performance to obtain a significant difference with only eleven pairs. As one can predict the direction of change where there is an expected difference, in this section we shall use a one-tail test where appropriate. In each case the null hypothesis will be that the control students give more responses in the first column of the given table than the experimental students.

## The graph of y=abs(x) at the origin

Of the matched pairs with and without calculus, those responding that there were no gradient are as in table 12.32:

Without previous calculus

| | no gradient | other |
|---|---|---|
| Experimental | 10 | 1 |
| Control | 1 | 10 |

($X^2$=4.32, significant at 5% level, 1 tail)

With previous calculus

| | no gradient | other |
|---|---|---|
| Experimental | 22 | 2 |
| Control | 12 | 12 |

($X^2$=8.17, significant at 1% level, 1 tail)

Table 12.32

There is a significant improvement in both groups of experimental students in their response that the graph has no gradient at the origin.

Those responding that there is no tangent were as in table 12.33:

Without previous calculus

| | no tangent | other |
|---|---|---|
| Experimental | 7 | 4 |
| Control | 4 | 7 |

($X^2$=0.73, not significant)

With previous calculus

| | no tangent | other |
|---|---|---|
| Experimental | 23 | 1 |
| Control | 17 | 7 |

($X^2$=3.75, significant at 5% level, 1 tail)

Table 12.33

Here the difference in the first table is not significant because of the Kenilworth experimental students who respond that there are two tangents. As we remarked earlier, there is an ambiguity here that some may see that "two tangents" means a left-tangent and a right tangent, but no single tangent.

The improvement in the second table is significant. However, the change in the control responses at first surprised me. I had initially thought that the students would be more likely to think that there was a tangent than that there was a gradient, yet fourteen (in total) said there were one or more tangents, whilst twenty two asserted that there were one or more gradients. Looking more closely at the responses showed that there were eleven students giving "balance" tangents, one with two tangents (left and right) and two with many tangents. The gradients were eleven "balance" gradients, six right gradients, two responding

$\pm 1$ and three others. The main difference is contributed by the five right gradients and three others. Invariably these students tried to calculate the gradient by differentiating the formula and obtained an erroneous answer, for example differentiating abs(x) to get abs(1)=1. Although they obtain the value +1, it may be the result of an incorrect calculation rather than a true right gradient.

One may conjecture that some students obtain a tangent by *looking* and a gradient by *calculating*, in this case leading to different answers.

The graph of $y=\sqrt{(abs(x))}$ at the origin

The responses for this graph were earlier grouped in two different ways. The first considered those saying "no gradient" (or "no tangent") compared with other responses. Table 12.34 gives the classification for those saying "no gradient":

Without previous calculus

| | no gradient | other |
|---|---|---|
| Experimental | 9 | 2 |
| Control | 1 | 10 |

($\chi^2$=8.98, significant at 1% level, 1 tail)

With previous calculus

| | no gradient | other |
|---|---|---|
| Experimental | 19 | 5 |
| Control | 11 | 13 |

($\chi^2$=4.36, significant at 5% level, 1 tail)

Table 12.34

Both groups of experimental students give significantly more responses in the "no gradient" category.

An analogous classification for those responding that there is no tangent is given in table 12.35:

Without previous calculus

| | no tangent | other |
|---|---|---|
| Experimental | 9 | 2 |
| Control | 2 | 9 |

($X^2$=6.56, significant at 1% level, 1 tail)

With previous calculus

| | no tangent | other |
|---|---|---|
| Experimental | 21 | 3 |
| Control | 13 | 11 |

($X^2$=4.94, significant at 5% level, 1 tail)

Table 12.35

A second classification for this example groups together both "no gradient" and "infinite gradient" (or "no tangent" and "infinite tangent"), as both are acceptable mathematical responses. Table 12.36 gives those responding "no gradient" or "infinite gradient" compared with the rest:

Without previous calculus

| | none/∞ | other |
|---|---|---|
| Experimental | 11 | 0 |
| Control | 3 | 8 |

($X^2$=9.62, significant at 0.1% level, 1 tail)

With previous calculus

| | none/∞ | other |
|---|---|---|
| Experimental | 24 | 0 |
| Control | 14 | 10 |

($X^2$=10.23, significant at 0.1% level, 1 tail)

Table 12.36

Here we see that all the experimental students respond that the graph has no gradient or an infinite gradient, whereas a significant number of control students respond otherwise. The most common other response (given by six control students with previous calculus experience and six without) is that the gradient is zero (corresponding to a "balance" tangent).

The analogous classification for tangents, grouping together "no tangent" and "vertical tangent", is as in table 12.37:

Without previous calculus

| | none/vertical | other |
|---|---|---|
| Experimental | 11 | 0 |
| Control | 7 | 4 |

($X^2$=2.75, (just) significant at the 5% level, 1 tail)

With previous calculus

| | none/vertical | other |
|---|---|---|
| Experimental | 22 | 2 |
| Control | 17 | 7 |

($X^2$=2.19, not significant)

Table 12.37

Although the experimental students show more responses in the "none/vertical" category than the controls, the difference is less dramatic than all previous categories, reflecting the degree of conflict created by the question. The statistics are measuring the comparative minorities of students giving responses in categories other than "none" or "vertical". In total the experimental students only give two "other" responses: one seeing "many" tangents and the other a "balance" tangent. There are eleven "other" responses from the control students: two seeing "many" tangents, five "balance" tangents, and two others drawing a single tangent by eye that seems to touch the cusp on one side at a slight angle to the vertical.

Although one result in table 12.37 is (just) significant at the 5% level and the other is (just) not, there is clearly no statistical difference between the performance of the two experimental groups without previous calculus (11:0) and with (22::2).

<u>The graph of y=abs($x^3$) at the origin</u>

We saw earlier that there is no significant difference between experimental and control in drawing the tangent to this graph at the origin. The only significant difference is that more control students calculate the gradient using an erroneous derivative, such as abs($3x^2$). Comparing those giving the correct gradient, without using an incorrect formula, against those giving another response (either using an incorrect formula or giving an unsatisfactory response) gives table 12.38:

Without previous calculus

| | gradient=0 | other |
|---|---|---|
| Experimental | 10 | 1 |
| Control | 9 | 2 |

($X^2$=0, not significant)

With previous calculus

| | gradient=0 | other |
|---|---|---|
| Experimental | 21 | 3 |
| Control | 13 | 11 |

($X^2$=4.94, significant at the 5% level)

Table 12.38

Here there is a difference between those with, and those without, previous calculus. Comparing the experimental students in the first table (10:1) and those in the second (21:3), there is clearly no significant difference in their performance ($X^2$=0.08). Although there is a visible difference between the controls in the first table (9:2) and those in the second (13:11), it is not statistically significant ($X^2$=1.43).

What seems to have happened here is that the controls without previous calculus experience include a majority who have written down the correct result without showing any working. We cannot be sure whether they have used a correct method, or an incorrect method that they have not written down.

The important factor is that there is no significant difference on this question between the experimental students with, and those without, calculus experience.

The graph of $y= \begin{cases} x & (x\leq 0) \\ x+x^2 & (x\geq 0) \end{cases}$ at the origin

The numbers of students specifying the gradient to be 1, as compared with other responses are given in table 12.39:

Without previous calculus

| | gradient=1 | other |
|---|---|---|
| Experimental | 11 | 0 |
| Control | 7 | 4 |

($\chi^2$=2.75, (just) significant at the 5% level, 1 tail)

With previous calculus

| | gradient=1 | other |
|---|---|---|
| Experimental | 23 | 1 |
| Control | 13 | 11 |

($\chi^2$=9.00 significant at the 1% level)

Table 12.39

Although there is a difference between the significance levels, there is clearly no significant difference between the experimental students in the first table (11:0) and those in the second (23:1).

The numbers drawing the standard tangent, as compared with other responses including the generic tangent, are given in table 12.40:

Without previous calculus

| | standard | other |
|---|---|---|
| Experimental | 7 | 4 |
| Control | 2 | 9 |

($X^2$=3.01, significant at the 5% level, 1 tail)

With previous calculus

| | standard | other |
|---|---|---|
| Experimental | 17 | 7 |
| Control | 11 | 13 |

($X^2$=2.14 not significant)

Table 12.40

Again, although there is a difference in the significance levels, there is no statistically significant difference between the two experimental groups. (The class numbers 7:4 and 17:7 give $X^2$=0.04.)

The graph of $y= \begin{cases} x & (x\leq 0) \\ x^2 & (x\geq 0) \end{cases}$ at the origin

This last example, with a function given by two different formulae having different gradients on either side of the point under consideration is a little more problematic in interpretation. Comparing those saying "NO" to the question "can you calculate the gradient at $x=0$?", to others gives table 12.41:

Without previous calculus

| | NO | other |
|---|---|---|
| Experimental | 8 | 3 |
| Control | 7 | 4 |

($X^2$=0, not significant)

With previous calculus

| | NO | other |
|---|---|---|
| Experimental | 22 | 2 |
| Control | 13 | 11 |

($X^2$=9.00 significant at the 1% level)

Table 12.41

As we saw earlier, the "NO" category includes both those that know why there is no (single) gradient and those who do not know what to do, marring the comparison. We saw that the experimental students responding "NO" show more understanding in their responses. Comparing the two groups (8:3 and 22:2), the difference is not statistically significant ($X^2$=0.93).

Finally, table 12.42 compares those asserting that there is no tangent, compared with all other responses:

Without previous calculus

| | no tangent | other |
|---|---|---|
| Experimental | 11 | 0 |
| Control | 6 | 5 |

($X^2$=4.14, significant at the 5% level, 1 tail)

With previous calculus

| | no tangent | other |
|---|---|---|
| Experimental | 21 | 3 |
| Control | 18 | 6 |

($X^2$=0.55, not significant)

Table 12.42

## Comment on the matched group comparisons

In every comparison in the previous section we have seen that each experimental group has the advantage over the corresponding control group in terms of the given analysis. Eighteen of the twenty two comparisons made between experimental and control are statistically significant, the exceptions being those in tables 12.33 (tangent), 12.37 (tangent*), 12.38 (gradient), 12.40 (tangent*), 12.41 (gradient), 12.42 (tangent*), where the asterisk denotes the two cases with a non-significant difference occurring between those with previous calculus experience. Thus both those with previous calculus experience, and those without, show eight out of eleven tables with statistically significant improvements for experimental over control.

In the cases without statistical significance the distributions for the two experimental groups are as follows (where the asterisk denotes the group with previous calculus experience):

| | | | |
|---|---|---|---|
| Table 12.33 tangent : | E | 7 | 4 |
| | E* | 23 | 1 |
| Table 12.37 tangent : | E | 11 | 0 |
| | E* | 22 | 2 |
| Table 12.38 gradient: | E | 10 | 1 |
| | E* | 21 | 3 |
| Table 12.40 tangent : | E | 7 | 4 |
| | E* | 17 | 7 |

| | | | |
|---|---|---|---|
| Table 12.41 gradient: | E | 8 | 3 |
| | E* | 22 | 2 |
| Table 12.42 tangent : | E | 11 | 0 |
| | E* | 21 | 3 |

In each case, with the null hypothesis that the groups E,E* have the same proportions, the respective values of $X^2$ are 4.02, 0.41, 0.08, 0.00, 0.93, 0.33. Only the first is statistically significant.

Why? Looking at the diaries, the reason becomes fairly evident. Only one session (page 184 *et seq.*) was taken up with the discussion, just before half term, and it was not followed up in any detail. Indeed, on looking back to the summary of the Kenilworth experience which I wrote at the time, nearly twelve months before I wrote this chapter, I find myself saying (on page 202):

> Limited time meant that hoped-for study of concepts, such as further discussion of tangents, or of the dy/dx notation, or more examples of non-differentiable functions, were squeezed out. The need to get through the required number of techniques, to encounter all variations of difficulties that might occur in formal manipulation on the examination paper, proved extremely strong.

Apart from this lack of clarity over whether distinct left and right tangents should be counted as two or none, there is no evidence from these statistics to show that using the computer approach has different effects on those with or without previous calculus. On the contrary, in all cases it gives a statistically significant improvement for experimental over control in eight out of eleven comparisons for both groups.

## Pandora's Box

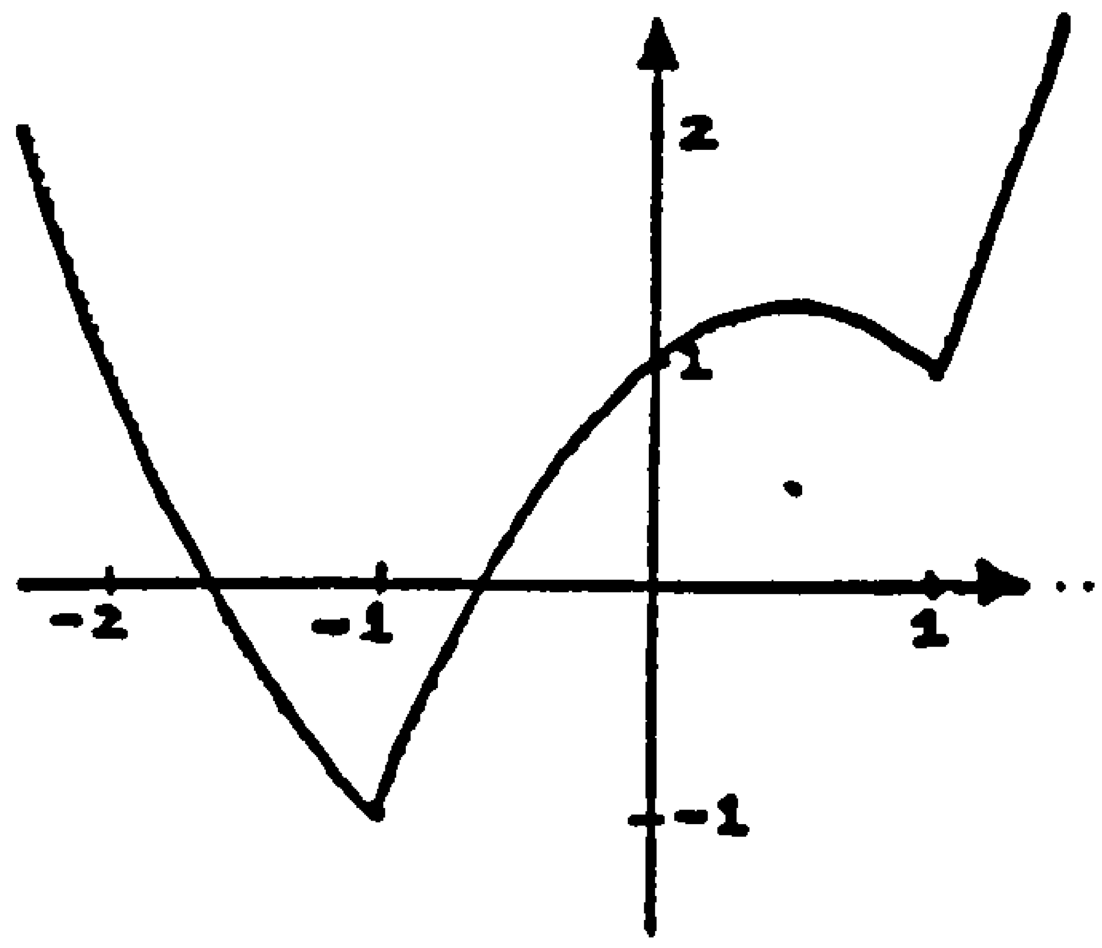
So far we have been comparing responses to questions involving gradient and tangent, without explicit mention of the derivative. The final task on the tangent investigation looked at a graph with different left and right derivatives at a point and asked the students how many gradients, derivatives, and tangents the graph had there (figure 12.43).

8. **The displayed picture is the graph of y=abs(x²-1)+x. It can also be expressed as: y=x²-1+x if x≤-1 or x≥1 and y=1-x²+x for -1≤x≤1.**

Which of the following is true:
(a) the graph has no gradient at x=1
(b) the graph has one gradient at x=1
(c) it has two gradients at x=1
(d) it has more than two gradients at x=1
(e) other comment (specify).....

Circle one of: a - b - c - d - e
If your response is (a), say why not, if (b),(c) or (d) specify the gradient(s):

How sure are you of your answer? (underline one)
Certain/fairly sure/fairly doubtful/very doubtful.




\##############################################
Which of the following is true?
(a) the graph has no derivative at x=1
(b) the graph has one derivative at x=1
(c) the graph has two derivatives at x=1
(d) the graph has more than two derivatives at x=1
(e) other comment (specify).....

Circle one of: a - b - c - d - e

If your response is (a), say why not, if (b),(c) or (d) specify the derivative(s):

How sure are you of your answer? (underline one)
Certain/fairly sure/fairly doubtful/very doubtful.

\##############################################
Which of the following is true:
(a) the graph has no tangent at x=1
(b) the graph has one tangent at x=1
(c) the graph has two tangents at x=1
(d) the graph has more than two tangents at x=1
(e) other comment (specify).....

Circle one of: a - b - c - d - e

If your response is (a), say why not, if (b) (c) or (d) draw the tangent(s) on the above graph...

How sure are you of your answer? (underline one)
Certain/fairly sure/fairly doubtful/very doubtful.

Figure 12.43

One may hope that the experimental students would reply coherently and consistently to all three notions. But we already know that the Kenilworth experimental students were unclear as to whether distinct left and right tangents are to be counted as two tangents or none. The responses to the questions from the experimental students were as in table 12.44, where G,D and T refer to gradients, derivatives and tangents, and the entries are either the given number (0,1 or 2), the symbol ∞ denoting "many" or "an infinite number" or a question mark denoting some other response. In later tables there also appears a dash to denote no response at all.

| KE (N=14) | | | | BE1 (N=12) | | | | BE2 (N=16) | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | G | D | T | | G | D | T | | G | D | T |
| KE01 | 2 | 2 | 0 | BE101* | 1 | 1 | 1 | BE201* | 0 | 0 | 0 |
| KE02 | 0 | 0 | 0 | BE102* | 0 | 0 | 0 | BE202* | 0 | 0 | 0 |
| KE03 | 0 | 1 | 0 | BE103* | 0 | 0 | 0 | BE203* | 0 | 0 | 0 |
| KE05 | 2 | 1 | 2 | BE104* | 0 | 0 | 0 | BE204* | 0 | 0 | 0 |
| KE06 | 0 | 0 | 0 | BE105* | 0 | 0 | 0 | BE205* | 0 | 0 | 0 |
| KE07* | 2 | 1 | 2 | BE107* | 0 | 0 | 0 | BE206* | 0 | 0 | 1 |
| KE08 | ∞ | ∞ | ∞ | BE108* | 0 | 0 | 0 | BE207* | 0 | 0 | ∞ |
| KE09 | 0 | ? | 0 | BE109* | 0 | 0 | 0 | BE208* | 0 | 0 | ∞ |
| KE11 | 0 | 0 | 0 | BE110* | 0 | 1 | 0 | BE209* | 0 | 0 | 0 |
| KE12 | 0 | 2 | 0 | BE111* | 0 | 0 | 0 | BE210* | 0 | 0 | 0 |
| KE13 | 0 | 0 | 0 | BE112* | 0 | 0 | 0 | BE211* | 0 | 0 | 0 |
| KE14 | 0 | 0 | 0 | BE113* | 2 | 2 | 0 | BE212* | 0 | 0 | 0 |
| KE15* | 2 | 2 | 2 | | | | | BE213* | 0 | 0 | 0 |
| KE16* | 2 | 0 | 0 | | | | | BE214* | 0 | 0 | 0 |
| | | | | | | | | BE215* | 0 | 0 | 0 |
| | | | | | | | | BE216* | 0 | 0 | 0 |

Table 12.44

The table shows that, although I failed to produce a fully coherent result at Kenilworth, the other two teachers were significantly better with their classes. Referring back to chapter eight it quite clearly shows that the discussions we had at Kenilworth on the relationship between gradient, derivative and tangent (reported on page 185 *et seq.*) hardly mentioned the derivative at all. If one deletes the middle column of the Kenilworth table, then one sees that there is a consistency between the first and last columns, except for KE01 and KE16* who both see 2 gradients, but no tangent. These students see a left-gradient and a right-gradient, and deduce that the graph has no derivative or tangent.

Every other student, bar one, sees the same number of gradients and tangents (either none, or two, the latter being the left and right concepts). Six see no gradient or tangent, and three mention two gradients and tangents. The exception is KE08, who holds to an earlier concept image and mentions an infinite number of gradients and tangents.

The confusion on this one point at Kenilworth was sown in a single lesson just as the half term holiday gave a twelve day break before the calculus was studied again. After the break we followed a standard calculus course with "smooth curves", mainly polynomials, supported by computer graphics. The very theory that I have put forward suggests that I would fail. The students' thinking would be affected by the actual work they do, so that the concept image that grew more dominant was that of "smooth curves". They were still able to evoke some of the notions of non-differentiability when required, but we never properly sorted out this part of the theory.

Meanwhile, the diaries of the teachers at Barton Peverill showed a somewhat different story. They followed the plan I had given them (Appendix 1), and table 12.44 shows how successful they were. In BE1 and BE2, only three students in each case failed to specify the response "0 0 0", and one of these, BE101*, misread the question, thinking it referred to the point y=1, rather than x=1.

The degree of the success becomes more apparent when one looks at comparable responses from the control groups (table 12.45):

KC (N=9)

| | G | D | T |
|---|---|---|---|
| KC01 | 0 | 1 | 2 |
| KC02 | 1 | 2 | 0 |
| KC03* | 1 | 1 | 1 |
| KC04 | 3 | 2 | 3 |
| KC05* | 0 | 2 | 1 |
| KC06 | 1 | 2 | 1 |
| KC07 | 1 | 2 | 0 |
| KC08 | 0 | 1 | 2 |
| KC09 | 2 | ? | 2 |

BC1 (N=13)

| | G | D | T |
|---|---|---|---|
| BC101* | 2 | 2 | 2 |
| BC103* | 0 | 2 | 0 |
| BC104* | 0 | 1 | 2 |
| BC105* | 1 | ∞ | 2 |
| BC106* | ∞ | ∞ | ∞ |
| BC108* | 1 | 1 | 1 |
| BC109 | 0 | 1 | 2 |
| BC110* | 1 | 2 | 1 |
| BC111* | 0 | 0 | 0 |
| BC112* | 1 | 1 | 1 |
| BC113* | 2 | ? | 1 |
| BC114 | 0 | 0 | 0 |
| BC115 | 2 | 2 | 2 |

BC2 (N=17)

| | G | D | T |
|---|---|---|---|
| BC201* | 2 | 2 | 2 |
| BC202* | 2 | 2 | 2 |
| BC203* | 2 | 2 | 0 |
| BC204* | 0 | 1 | 0 |
| BC205* | 0 | 0 | 0 |
| BC206* | 0 | 1 | 1 |
| BC207* | 0 | 0 | 0 |
| BC210* | ∞ | 2 | ∞ |
| BC211* | 2 | 2 | 2 |
| BC212* | ∞ | 1 | ∞ |
| BC213* | 0 | 0 | 0 |
| BC214* | 2 | 1 | 2 |
| BC215* | 1 | 2 | ∞ |
| BC216* | 0 | 1 | 0 |
| BC217* | 0 | 2 | 0 |
| BC218* | 2 | 2 | 2 |
| BC219* | 2 | 1 | 2 |

BC3 (N=13)

| | G | D | T |
|---|---|---|---|
| BC302 | 1 | 2 | 1 |
| BC303 | 1 | 1 | 0 |
| BC304 | 0 | ? | 1 |
| BC305* | 1 | 1 | ∞ |
| BC306* | 2 | 2 | 2 |
| BC308* | 1 | 2 | 0 |
| BC309* | 2 | 2 | 1 |
| BC310* | 0 | 0 | 0 |
| BC311* | 1 | 1 | 1 |
| BC313* | 2 | ? | 2 |
| BC315* | 1 | 1 | ? |
| BC316* | 1 | 0 | 1 |
| BC318* | 1 | 1 | 1 |

BC4 (N=11)

| | G | D | T |
|---|---|---|---|
| BC401* | 2 | 1 | 2 |
| BC402* | 0 | 0 | 0 |
| BC403* | 0 | 0 | 0 |
| BC404* | 2 | 0 | 2 |
| BC405* | 0 | 0 | 0 |
| BC406* | 2 | 2 | 0 |
| BC407* | 0 | 0 | 0 |
| BC408* | 0 | 0 | 0 |
| BC409* | 2 | 2 | ∞ |
| BC410* | 0 | 1 | 2 |
| BC412* | 2 | 2 | 0 |

Table 12.45

Amazingly, table 12.45 shows that in each control class, there are almost as many different responses as there are students (table 12.46).

| | students | different responses |
|---|---|---|
| KE | 14 | 9 |
| BE1 | 11 | 3 |
| BE2 | 16 | 4 |
| KC | 9 | 8 |
| BC1 | 13 | 9 |
| BC2 | 17 | 10 |
| BC3 | 13 | 13 |
| BC4 | 11 | 8 |

Table 12.46

The different responses arise in various ways. For example a student may consider that there is no gradient because it is undefined, one gradient because it is a left, right or "average", or because only one formula is differentiated and the other neglected. Two gradients usually arise because both left and right are calculated (though not necessarily correctly) and the student may consider there are many gradients because (s)he sees an infinite number of tangents.

The derivatives cause a different problem. There may be no derivative because the student knows the left and right derivatives are different as formulae or as specific numbers, or simply because (s)he doesn't know how to calculate it. There may be one derivative because only one formula is used, or two derivatives because both are differentiated, or many (unspecified) derivatives because the student sees an infinite number of tangents.

Finally the tangents exhibit just as many possibilities: none because the graph is not smooth, or because it has a corner. There may be one tangent drawn as a "balance" tangent or, less often, a single left or right tangent. The "balance" tangents are of two different kinds, some drawn horizontal, others at an angle so as to pass through the other corner on the curve. When the response is two tangents, these are invariably seen as the left and right tangents. More than two are because the student sees many possibilities that touch the curve at the single point.

Coupled with these differences is the possibility that a given student may see different features in combination: two gradients, two derivatives, but no tangent, and so on... The possibilities are many and varied.

Even by the time the students have completed their sixth form course and a more able subset go on to study mathematics at university, the difficulties are only partially resolved (table 12.47):

U2 (N=57)

| | G | D | T | | G | D | T | | G | D | T |
|---|---|---|---|---|---|---|---|---|---|---|---|
| U50 | 0 | 2 | 0 | U66 | 0 | 0 | 2 | U82 | 0 | 0 | 0 |
| U51 | 0 | 0 | 0 | U67 | 0 | 0 | 0 | U83 | 0 | 2 | 0 |
| U52 | 2 | 2 | 2 | U68 | 0 | 0 | 0 | U84 | ∞ | 2 | ∞ |
| U53 | 0 | 0 | 0 | U69 | 0 | 0 | 0 | U85 | 2 | 2 | 2 |
| U54 | 2 | 2 | 0 | U70 | 0 | 2 | 2 | U86 | 2 | 2 | 2 |
| U55 | 2 | 2 | 2 | U71 | 2 | 2 | 2 | U87 | 0 | 2 | 0 |
| U56 | 0 | 0 | 0 | U72 | 2 | 2 | 2 | U88 | 0 | 0 | 0 |
| U57 | ? | ? | ? | U73 | 2 | 2 | 2 | U89 | 0 | 0 | 0 |
| U58 | – | – | – | U74 | 2 | 2 | 1 | U90 | 1 | 2 | 1 |
| U59 | 0 | 0 | ∞ | U75 | 1 | 1 | 1 | U91 | 0 | 0 | 0 |
| U60 | 0 | 0 | 0 | U76 | – | – | – | U92 | 2 | 2 | 0 |
| U61 | – | – | – | U77 | – | – | – | U93 | 2 | 0 | 2 |
| U62 | 0 | 0 | 0 | U78 | 2 | 1 | 0 | U94 | 2 | 2 | 2 |
| U63 | 0 | 2 | 0 | U79 | ∞ | 2 | ∞ | U95 | 0 | 0 | 0 |
| U64 | 0 | 0 | 0 | U80 | 0 | 2 | 0 | U96 | 0 | 0 | 0 |
| U65 | 0 | 2 | 0 | U81 | 0 | 2 | 0 | | | | |

Table 12.47

Even here, where there is a settling down to three favourite choices:

14 responses saying 0 0 0

8 saying 2 2 2

and

6 saying 0 2 0,

there are still fourteen different choices in all.

Thus teachers following traditional courses who are concerned with the difficulties of left and right derivatives have every reason to feel justified by their fears. If too brief a period is spent on counterexamples, as in the group KE, then there may seem to be little improvement in the ability to deal with cases where the left and right gradients differ. However, this must be seen in the light of the rest of the results in this thesis where the group KE has clearly benefited from their ability to visualize gradients. The performance of groups BE1 and BE2 in this respect show that, given care and attention, a fuller understanding is possible.

## Summary

The naive definition of the tangent, that it "touches only one point of the curve", can have a coercive effect on students, leading them to draw a "generic" tangent that touches only once in a case where the true tangent touches more than one point. This persists through to a proportion of the best students at university. Those students using the generic organisers on the computer were less prone to this interpretation and significantly more experimental students gave the correct response.

The experimental students, who had discussed examples and non-examples of the concepts of gradients, tangents and derivatives, were able to give a more coherent response to all the questions than the controls. A small proportion of the latter visualized "balance tangents" and calculated "average gradients" at points on curves with different left and right gradients. They were also more likely to calculate gradients by differentiation than by eye, with a significant proportion obtaining the correct gradient of abs($x^3$) at the origin by an incorrect method of differentation. In addition they experienced difficulties in knowing what to do with a function made up by different formulae on different parts of the domain.

The most noticeable flaw in the experimental students' interpretations arose in the Kenilworth students' confusion over the formal decision to say that a point where left and right gradients were different should be considered to have no gradient rather than two gradients. Apart from this the experimental students were able to use their ideas in new situations, in particular where the function was given by different formulae on either side of the point under consideration.

The evidence in this chapter supports the hypothesis that the generic organiser GRADIENT offers the facility to discuss the question of left and right gradients in a simple manner. Without such a discussion, the control students' explanations of what happens at a point with different left and right derivatives cover a wide spectrum of interpretations.

# IV

# Conclusions

# 13. Conclusions and suggestions for further research

*To be able to reflect requires an ability to*
*perceive*
*recognise*
*articulate*
*and assimilate*
*what actually happened, without judgement*
*and without embroidery.*

Mason,
Burton,
& Stacey
[1982]

In this thesis I have effectively been both advocate and judge, building up a theory, putting forward its case as persuasively as possible, and then testing it to see if it works. But the readers of this thesis will be the jury, and in this chapter I present a summary of the case for the jury to make its judgement.

## Building the theory

It is my contention that the mathematician's way of constructing a curriculum for learning higher level mathematics is based more on an analysis of the mathematics itself, than on the cognitive demands facing the learner. If instead one attempts to develop the mathematical theory appropriate for learning, taking into account the growing conceptual imagery of the student, one arrives at a *cognitive* approach to the subject. This first requires research into the cognitive difficulties facing the learner in understanding the mathematics under consideration and then a rethink of the mathematical development in a manner that is both cognitively and mathematically appropriate.

Ausubel [1978] puts forward the notion of an *advance organiser* to give the learner an overview of the knowledge domain to be presented. An advance organiser must relate both to the current cognitive structure of the learner and to the knowledge domain. Thus it presupposes that the learner has appropriate knowledge to enable him to appreciate an overview.

There may be times when the learner does not have the appropriate knowledge available to him. Prefacing Ausubel et al. [1978] is the advice:

> *If I had to reduce all of educational psychology to just one principle, I would say this:*
> *The most important single factor influencing learning is what the learner already knows.*
> *Ascertain this and teach him accordingly.*

Therefore, instead of using advance organisers to teach beginning calculus I propose a complementary principle, based on the work of Dienes: the notion of a *generic organiser*. This is an environment which enables the learner to manipulate *examples* of a specific mathematical concept or related system of concepts, thus focussing the learner's attention on the salient features of the more general abstract concept embodied in the examples. By using generic organisers, the learner gains a *sense* of the abstract concept and his mind becomes more ready to assimilate the higher level ideas.

A generic organiser has the advantage that it lowers the cognitive demand for the learner. In the terms proposed by Skemp [1979], it allows the concepts to be built and tested in mode 1, through direct manipulation and experiment, instead of the normal modes of mathematical theory building and testing through transmission of more abstract concepts by communication and discussion, or through deductions within ones own mind using the internal consistency of the mathematics itself.

Although a generic organiser can be used for exploration and discovery learning, I contend that those developed for the computer using current technology are insufficient to enable the learner to develop the higher order concepts without the help and support of an *organising agent*, which today means a sympathetic teacher probably with accompanying texts or worksheets, although in the world of tomorrow one may envisage an organising agent at least partially provided by computer software.

My main thesis is that appropriately designed generic organisers, supported by an appropriate organising agent, can provided students with global gestalts for mathematical processes and concepts at an earlier stage of their development than occurs with current teaching methods.

The initial learning of the calculus is often referred to as an *intuitive* approach, as compared with the *rigorous* development of analysis that is later available to the privileged few. I would contend that this approach is *not* intuitive in a psychological sense, by which I hypothesise that the appropriate intuitions for the ideas are not already present in the mind of the learner. On the contrary, research of Schwarzenberger & Tall [1978], Tall & Vinner [1981], Cornu [1981,1983], Robert [1982] has shown that there are underlying obstacles to the students' understanding of the subject. In this thesis we also see (pages 290,291) that the idea of a chord approaching a tangent is not psychologically "intuitive" in the sense that it is an a *priori* spontaneous method of solution. Nor is much of the language used in the "intuitive" approach conducive to evoking the appropriate mathematical concepts (see chapter 11).

The long-term goal of a cognitive approach to the calculus, using generic organisers, is to provide those intuitions which are both relevant as an end in themsleves and also as a basis for a later formal theory of analysis. The aim is to develop in the learner a cognitive structure that is consonant with the formal theory, not in opposition to it.

Of course such a high-born aim may not be achieved. For, as

Piaget has shown, at each stage of our development we seek to achieve an equilibrium with our environment, which means an ability to cope with the variety of situations which are presented to us in a flexible and stable manner. Thus we develop techniques appropriate for our current environment which may not extend to a wider context. Later, when we meet new problems, our old cognitive structure may no longer be appropriate and a conflict arises whose resolution requires cognitive reconstruction.

Cornu [1981,1983] has used Bachelard's theory of *cognitive obstacles* to describe how knowledge may be satisfactory for solving problems in a certain context and, precisely because it is so successful, it becomes anchored in the mind, only to become an obstacle in a different situation. Tall & Vinner [1981] describe some of my early research where I extended Vinner's notion of *concept image* and *concept definition* to explain certain conflicts that occur in individuals' minds.

The concept image is the total cognitive structure associated with the concept. Only parts of it are evoked at any given time and these evoked images may have aspects that prove conflicting should they be evoked simultaneously. The concept definition is a form of words used to specify the concept. In practice it is the concept image, generated by experience and internal processing, that governs the individual's thinking. This experience may unwittingly produce conflict in dealing with the formal structure of the mathematics. For instance, vernacular use of terms such as "limit", "tends to", "approaches", "converges" may subtly affect the student's interpretation of the formal ideas.

Generic organisers are intended to fulfil the role of enabling the learner to operate within a limited environment that focuses on a given concept. The learner develops an equilibrium with that environment and is able to operate within it, developing a concept image appropriate for the current task. A good generic organiser should also contain within it the seeds of ideas that lead on to a higher level where reconstruction may be necessary, although they may not be apparent to the learner at the time. It is with this possibility in mind that the generic organisers in "Graphic Calculus" were designed, containing the seeds of ideas for later developments in mathematical analysis.

To do this requires an analysis of the mathematical concepts, from mathematical cultural, and cognitive viewpoints. It is necessary to understand not only the logic of the mathematics that is the product of painstaking theory-*testing*, but also the development of human culture responsible for *building* the potent combination of concepts in the current theory.

For this reason I analysed both the cultural development and the various possible approaches to the calculus that are available now, or may become available in the near future.

It is clear that we are in a state of flux with the coming of the computer. One may conjecture that we are living in the time of a Kuhnian change in paradigm in which crystal-gazing into the future is subject to many stresses that make it unpredictable.

I have argued that the coming of new technology may very well change the importance of the place of calculus in the curriculum and the necessity to practice all the intricate skills of differentiation and integration. Symbolic manipulators will carry out the calculations and may relieve us of the tedium of manipulation in much the same way as calculators relieve us of the necessity to practice long division. However, if this transpires, it makes it *more* important to understand the underlying ideas of rate of change (differentiation) and cumulative growth (integration), not less.

I decided that the generic organisers should provide intuitions for the formal theories of both standard and non-standard analysis, although both would be *implicit*, rather than explicit. Thus the ordinary teacher could use the organisers without being aware of the formal theories and still provide a useful learning experience for the students. What is important is that the generic organisers are used, not only to provide *examples* of the concepts, but also *non-examples*, to give a fuller concept image.

The generic organisers developed start with the notion that small portions of some graphs "look straight" under high magnification.

Thus the gradient of the graph at a point is the gradient of the almost straight part under magnification. The program GRADIENT provides an environment to explore both the limiting process calculating the gradient of a chord AB as B moves towards a fixed point A, and a dynamic gestalt of the approximate global gradient traced as an extended chord steps along the graph.

One may hypothesise that the use of such dynamic interactive programs provides a meaningful visualization that is not possible in a static text-book, nor even in a video film that one cannot use to explore different examples at will.

Further organisers consider the dynamic development of the notion of area under a graph, the antiderivative of a function (finding the graph of a function knowing its gradient), moving on to programs drawing numerical solutions of differential equations. These organisers support a theory that builds up to the notion of a differentiable manifold as a subset of n-dimensional space that is "locally flat".

One may show that the organisers are part of an appropriate *mathematical* theory by checking the theory's internal consistency. But the question of whether it provides a suitable *cognitive* approach may only be tested by empirical research.

## Testing

The testing, as described in this thesis is limited to the study of the generic organisers for differentiation. These are the programs to MAGNIFY a graph and to display its GRADIENT. (The program BLANCMANGE was also used at the same time to provide a graphic example of an everywhere continuous, nowhere differentiable function.)

The main thrust tested the effects of the program GRADIENT. This was used in three separate classrooms, one where I took an active part, and two others where teachers supplemented their normal work by activities using the computer, following my suggestions (appendix 1). We all succeeded to a certain extent, and we all failed.

In the Kenilworth control group KE, I failed to make it clear that a graph with a corner having a left-gradient and a right-gradient was considered mathematically to have *no* gradient, rather than two. The other two classes succeeded on this score.

In the Barton Peverill group BE1, four out of twelve students performed badly on a test to sketch the gradient of given graphs, in the sense that they scored less than half marks on the task. Only one student out of the other two experimental groups scored at such a low level.

The Barton Peverill group BE2 failed on the specification of a function which was not differentiable at a given point, whilst the other two groups performed somewhat better.

However, these three singular failures occur in a broad span of greater success on the tests, including the following tasks where the experimental students scored at a statistically significant higher level than the controls:

Sketching the derivative for a given graph, (Task (C), chapter 9).

Recognizing a derivative, (Task (E), chapter 9).

Specifying a non-differentiable function, (Task (F), chapter 9).

Responding more often to questions causing them to evoke the global gestalt of the gradient of the graph, (Tasks (H), (J), chapter 10).

Giving more dynamic or pre-dynamic responses to open-ended questions on differentiation from "first principles", the gradient of a graph, and the tangent to a graph at a point, (Tasks (G), (H), (I), chapter 10).

Relating the derivative to the gradient, and to the gradient function (Task (J), chapter 10)

One further task:

(A) Calculating numerical gradients from a picture (Chapter 9)

also showed a statistically significant improvement, but this occurred in a context where one of the items with negative x and y increments provoked a conflict amongst both experimental and control students and it would not be sensible to place too much reliance on the result.

At the same time, more traditional tasks showed no significant difference:

(B) differentiation from first principles (chapter 9)

(C) formal differentiation of polynomials and powers (chapter 9).

The improvements in the experimental groups are not brought about by the generic organisers alone. As the minor failures in each experimental group show, much of the success is due to the organisational agent, in this case the teacher who is directing the learning process. In a third school which agreed to take part, there was no significant improvement brought about by using the generic organisers, even though this school had a better provision of computers than either of Kenilworth or Barton Peverill. Here the students reported that they did not have enough time in using the programs (tables 8.5, 8.6), with individual comments reporting confusion (page 222) and lack of time and explanation (page 224). Clearly the role of the organising agent is of paramount importance in ensuring the success of a generic organiser.

Two factors emerged from the analyses of student responses at Kenilworth and Barton Peverill. First it became clear that, although experimental students gave more responses of a dynamic, or pre-dynamic, nature, very few students, either experimental or control, mentioned limiting processes, even though they had been discussed informally in all the classes. Second, it became noticeable that the control students often evoked the mental image of a tangent being a line through two very close points on the graph, despite the fact that efforts had been made to distinguish between a tangent and an extended chord through two close points.

To investigate the two phenomena more closely, an analysis was made of student written responses and of interviews in which these topics were discussed (chapter eleven). This confirmed the thesis of Cornu [1983], in that every class contained students having cognitive difficulties with the limit notion, often based on the vernacular use of the term, sometimes with the connotation that the limit is "never attained" or "never passed".

It is tragic therefore that the Examining Boards of the United Kingdom have jointly issued a declaration of the common core at advanced level founding the study of the calculus on:

> The idea of a limit and the derivative defined as a limit. The gradient of a tangent as the limit of the gradient of a chord. (GCE [1983])

The analysis in chapter eleven also showed that the phenomenon of believing a tangent to be a line through "two close points" on the graph is widespread in the control students too, the significant difference between experimental and control groups being that more experimental students believe it to be "true", whilst there is a broad spectrum of belief from "true" to "false" amongst the controls.

Counter to this belief is effect of the naive definition of a tangent as a line that "touches the curve at one point only", sometimes with the additional connotation that it "does not cross it". The latter concept image may be persistent and cause conflict with the formal concept definition at a later stage.

The final part of the testing of the theory in this thesis concerned the stability and the flexibility of the concept images of gradient, tangent, and derivative. What would happen if these were tested in a more general environment, where the function may not have a derivative at a point? Here two investigations probed the students' understanding of gradient, then tangent, in six different situations, starting with a simple case that could be solved by differentiation, moving through other cases with a "corner", a vertical cusp, a strange-looking formula that nevertheless had a derivative at the point in question, and two cases built up with different formulae on either side of the point, one which had equal left and right gradients, one which did not. The one with two formulae and equal left and right gradients also tested the effects of the naive definition of a tangent, as the graph on one side of the point was a straight line, so that the tangent touches the graph at more than one point.

The experimental students, with their concept image supported by the visualizations given by the examples and non-examples in the GRADIENT program, were able to handle the five more general cases significantly better than the controls. Even in the last two cases, presenting a totally new problem involving functions made up from two different formulae, they were well able to cope. There were still traces of the concept image of the naive definition of the tangent interfering with the way that some drew the tangent to a point where the graph was a straight line on one side. Instead of drawing the correct tangent coinciding with the straight line part, they drew a "generic" tangent which touched the curve at only one point. But there were significantly more control students who were beguiled into drawing such a generic tangent to the curve.

In the very last part of the questionnaire, asking the students to specify how many gradients, derivatives, or tangents a curve has at a corner, Pandora's box was opened. The control students gave a huge spectrum of different responses, but the two experimental groups BE1, BE2 who had investigated the notion of tangent using the computer, and had discussed the concepts with their teacher, were able to reply in a much more coherent manner.

The conclusions that I draw from this empirical work are that it was not perfect, but then human endeavour rarely is. It failed in small ways, but it succeeded in the broader aims of giving the students a global gestalt of the gradient of a graph that was able to transfer to more general contexts.

## Subsequent developments

A year has passed since the trials were completed. During that time the teachers have continued to work with their classes and I remain in close contact with Norman Blackett at Kenilworth. The students at Kenilworth went on to use the AREA programs for investigation and, with guidance, were able to conjecture the integral formulae for $x$, $x^2$, $x^3$ and $x^n$ and link it to the notion of differentiation to give an insight into the formal theory, as predicted in Tall [1986a].

Mr Blackett remains convinced that the students have a good mental image of the gradient concept, so much so, that he can indicate the ideas he is talking about by waving his hands in the air to discuss maxima and minima, without using the computer at all (Tall & Blackett [1986]).

The three experimental teachers are continuing to use the same methods with the next year's intake and others are taking up the methods and making them their own (see for instance, Kowszun's description of his own use of the program GRADIENT in Waddingham and Wigley (eds) [1985]).

The generic organisers have been used in lectures at university level in service courses for the teaching of calculus to biologists (using all the programs, including those on differential equations). They are also being used by mathematics lecturers for demonstration and student investigation in undergraduate analysis courses.

Thus the ideas are being taken into the culture. What remains is to see if they grow there...

## Suggestions for future research

The program of building and testing a cognitive approach to the calculus is yet to be completed. It still remains to take the ideas through to a higher level and to carry out research to test whether the concept images generated by the generic organisers are fully appropriate for later developments. At a time of such upheaval in the culture brought about by the new technology, it is difficult to carry out such long-term research in a properly controlled manner. Students from schools with so many different methods of teaching move on to a variety of different universities, each with their own preferred approaches to analysis. Meanwhile the technology moves on apace and the paradigm may change so as to render the research of lesser value.

There is much to do in relating the use of generic organisers to other approaches to the calculus, for example using either numerical methods, more general programming techniques or symbolic manipulation.

### Numerical Methods

This thesis suggests that a better cognitive approach to the tangent may be through the pre-dynamic idea of an extended chord (or secant) through two very close points on the graph. This is an operative method of finding a *practical* tangent which may later be developed into the limit concept to give the *theoretical* tangent. Indeed it may be sensible to precede any discussion on the calculus by a module of work in which the gradients of curves are calculated practically. This happens to a certain extent in physics and it would be valuable to see it carried through in a consistent way in mathematics also.

John Higgo, the chairman of the Mathematical Association's Committee on "The computer in the Mathematics Curriculum", has been taking this approach with less able groups of fifteen year olds preparing for O-level for two years now, preceding the usa of "Graphic Calculus" with the calculation of gradients using hand calculators.

The numerical methods option of the Oxford Local Examinations A-level has been modified at my suggestion to include the numerical calculation of gradients and areas in such a way that this can be integrated into the syllabus to follow the suggested cognitive approach. Now we may be able to test whether the theory works in broad practice.

Programming algorithms in the calculus

The Mathematical Association [1985] has produced a book and disc of "132 Short Programs", including a contribution I have written using a numerical approach to calculus concepts. Research is essential to see if the programming of algorithms helps or hinders the formation of mathematical concepts.

Fletcher [1983] has said:

> you know that you have really grasped a mathematical process if you can program a machine to perform it (page 18)

and employs this argument to suggest the introduction of short programs written by pupils to aid their mathematical understanding. At this stage he was referring to programming mainly in BASIC, with some references to other languages, such as Logo. BASIC is adequate for certain aspects of numerical algorithms, but is clearly inadequate for higher level mathematical processes such as symbolic manipulation.

The matter is contentious. Lane [1985] is very dismissive of programming as an aid to mathematics. Even if programming numerical methods prove to be useful for understanding the processes involved, the programming of the higher symbolic processes of mathematics may present a different picture.

Teachers have a great deal of expertise in teaching students to perform the processes of differentiation but few understand at the moment how these higher processes may be programmed as formal algorithms. Kowszun [private communication] has shown how to handle symbolic manipulation of polynomials in Logo, and is working on students carrying out the programming themselves, but it seems a considerably greater step for students to write algorithms for more general symbolic differentiation and integration.

It may be that the writing of a program has a higher cognitive demand than understanding the algorithm itself. Much depends on the availability of appropriate computer languages, for some are more suited for certain tasks than others.

Symbolic Manipulation

Considerable attention is being given, especially in North America, to the use of symbolic manipulation systems, such as MuMath, MACSYMA, and MAPLE, which allow formal manipulation of algebra, solution of equations, symbolic differentiation and integration, and related activities such as calculating the symbolic coefficients in Taylor's series. There is concern that, if the symbolic manipulation is taken over by a computer and not practised by the student, then the student may lack the experiences of carrying out fundamental processes that are an essential part of his mathematical growth. There is need for research here, both into the use of symbolic manipulators and into the relationship between such symbolic manipulation systems and a cognitive approach to calculus using computer graphics.

A more general task is to integrate the best features of the graphic approach with numerical, algorithmic and symbolic approaches to produce an appropriate cognitive development of the subject.

## Generic Organisers

Generic organisers may, in principle, be used in any part of mathematics. There is a need to build and test them in other areas. In Tall [1986b] a generic organiser is introduced to allow exploration of the concept of equivalent fractions for younger children. It has proved effective also with remedial pupils in later years of the secondary school.

The approach using generic organisers is widely applicable, and encourages a flexible use of computer technology where the same program may be used for teacher demonstration, discussion between teacher and pupil, between pupils themselves, or for investigation into the mathematical concepts to enrich the students' concept imagery.

## Cognitive Obstacles

In building cognitive approaches it is necessary to gain insight into students' cognitive difficulties. This is not an easy task. It is probably best done by a combination of clinical interviews with a few individuals and more widely distributed questionnaires. In the present thesis I realized that clinical interviews would be learning environments in themselves and therefore might distort research concerned with introducing generic organisers into the standard sixth-form classroom. More studies using clinical interviews are essential to gain insight into cognitive obstacles.

Although we know some of the difficulties caused, for example, by the limit concept, we are still not in a position to precisely tie down what is happening, and why. Nor do we fully understand the general process of maturation and how it takes place.

Development of new curricula

A long-term purpose of research into cognitive obstacles is to assist in the development of cognitive approaches to other areas in the mathematics curriculum. The process is a long and arduous one, yet is surely worthwhile. It requires an analysis of possible approaches to a knowledge domain to build a development that is cognitively appropriate and mathematically sound. This must build in stages so that each stage allows the learner to develop confidence in dealing with the necessary concepts at an appropriate level in a flexible and stable manner.

The curriculum builder and teacher must be aware of the cognitive obstacles that may occur and cause cognitive conflict when the context is broadened and the learner moves to the next stage. In such a development both advance organisers and generic organisers are of value, the former when the learner has the appropriate framework to be given an overview of the new task and the latter when building up higher order concepts by manipulating and exploring suitable examples.

The coming of the new technology has brought a new and exciting challenge to mathematics educators as we learn to develop it and bend it to our will. It helps us not only to carry out mathematical tasks, but also to support the learning process itself.

# Appendices

# Appendix 1

## NOTES FOR CALCULUS INVESTIGATIONS

### USING THE COMPUTER TO VISUALIZE THE CONCEPTS

The idea of the investigation is to see what differences occur when students use computer programs to see mathematical processes in action. Two (or more) comparable groups will be used. The control students will follow a standard course and the experimental students will follow the same course, supplemented by demonstrations and investigations using the programs in GRAPHIC CALCULUS.

Before the work on differentiation, both experimental and control groups should do the 3-page pre-test "INVESTIGATION INTO IDEAS OF THE CALCULUS".

It is assumed that the students will be working from Bostock & Chandler. The following notes are suggestions for the use of the computer programs. The control students should follow the text as it stands as closely as possible.

For the experimental group(s) the following specific teaching aims should be added (they should not be followed in the control groups):

1. The computer should be available at all lessons and used by the teacher for demonstration and

2. for the students for exploration whenever possible. (Suggestions will be given later as to how this may be done.)

3. In addition to the work in the text, the notion of GRADIENT should be emphasised as the gradient of the *graph* itself, by demonstrating that a differentiable graph highly magnified looks straight.

4. Examples of *non*-differentiable functions should be given to set the concept in context. These do not magnify to look straight (such as absx at the origin, abs($x^2$-1) at $x=-1$ or +1 or $2^{\text{absx}}$ at $x=0$).

5. The notion of tangent should be approached through using the computer to draw a line through two nearby points (using SUPERZOOM).

6. Examples of functions which do not have tangents should be given to set the concept in context (as in 3).

7. The link should be made that if a function is differentiable then it has a tangent and vice-versa.

The following suggestions are given to help achieve the above aims with the experimental group(s):

Bostock & Chandler, p.97: please note that the program GRADIENT uses the term "chord" to mean the full extended line through A,B, not just the segment AB. (The segment grows small as B moves to A, but the extended line tends to the tangent, so this concept is more appropriate.)

P.98: [1] Begin by using GRADIENT to show that a chord AB (usually) tends to a limit as B tends to A, and the gradient of the chord tends to a numerical limit. Try $y=x^2$ (with cursor movement to insert the power) from x=0 to 4, y=0 to 4 and select the chord option C to draw the chord through a=1, b=2, letting b tend to a. Also show what happens with a=1, b=0 as b tends to a.

What is the gradient of the graph at x=1 ? What is the gradient at x=1/2 ? Try drawing the graph from x=.499 to x=.501, taking option C for curve through the centre screen to show that highly magnified the graph looks straight. [3] What is the gradient of the curve here ? As the graph looks like a straight line highly magnified, we can approximate the gradient of the graph as the gradient of this line. Provided that two points are taken sufficiently close, the gradient of the graph is approximately equal to the gradient of the chord.

Feel free to experiment with other graphs, eg. f(x)=sqrx (x=-3 to 3, y=-3 to 3) or f(x)=1/x (same ranges) or f(x)=$2^x$ (x=-4 to 2, y=-1 to 5).

Can the students think of a function that does not have the property of looking straight under a magnifying glass ? [4] (They are hard to find in a form that can be typed into the program. A simple example is y=absx (absolute value of x) at the origin. (Try drawing it from x=-2 to x=2, option C for curve through centre.) Note that in this case the chord from the left has gradient -1 and from the right +1. Other nice examples include y=abs($x^2$-1) at x=-1 or x=1 or x=$2^{\mathrm{absx}}$ at x=0.

All these examples have odd points that don't magnify to lock straight. The BLANCMANGE program draws a function that is everywhere wiggly. If there is time show it, but don't do it if it has to be done hurriedly.

Do not neglect the formal aspects of differentiation (p.100), but you can check the calculation of the gradient of y=x(2x-1) at x=1 using GRADIENT. [1] The graph is a bit steep, so you'll need to get suitable ranges. (e.g. x=-5 to 5, y=-1 to 9.) [1]

p.101 use GRADIENT to draw y=1/x from -4 to 4, y=-4 to 4 and use the chord option C with a=2,b=3 to calculate the chord gradient as b moves to a. Particularly stress the negative gradient & also see what happens for a=2, b=1. [1]

p.102 Exercises [2]: Split the students up into numbered pairs. All the students do all the exercises, calculating the gradient formally using algebra. When pair n has finished exercise n, they draw the picture over a suitable range and use the chord option to calculate the gradient numerically. All students should look at the picture drawn.

At this stage I would like the students, in their own time, to do the investigation called "THE IDEA OF GRADIENT", calculating gradients on the computer & looking at a few cases where the left and right derivatives may not be equal. If it isn't possible for them to have individual access to the computer I have suggested a method of attack in class on the sheet "Administration of tests".

p.102 GRADIENT FUNCTION

[1] This section can be supported by option G in GRADIENT to draw the numerical approximation to the gradient function. $y=x(2x-1)$ is a bit steep to get a good drawing, so you might start with $y=x^2$ over the ranges $x=-2$ to 2, $y=-2$ to 2. Take option G and start with $c=1$ to get the idea, before repeating with $c=.1$ or $c=1/1000$. They may see that the gradient function approximates to $y=2x$.

Use the picture to emphasise the changing gradient of the graph, how it is *negative* to the left but getting less steep, zero gradient at the origin, then becoming positive and getting much steeper. After the gradient function has been drawn, try using the derivative option with $f'(x)=2x$ to compare with the derivative.

Next try Example 5a, $y=x(2x-1)$ ($x=-5$ to $x=5$, $y=-3$ to 7). Draw the gradient curve with a small value of c(=1/100) and use the derivative option to compare with $f'(x)=4x-1$...

Repeat the exercise with Example 5b. Encourage the students to suggest suitable ranges. (Don't be afraid to try a range and redraw the graph when you've got a better idea.)

p.104 Exercise 5b [2]: Repeat the format of exercise 5a, with the students typing in the functions, using the gradient option to draw the gradient curve, then the derivative option to input the formula they have found to see if it works. (A check on their accuracy!)

p.105 GENERAL DIFFERENTIATION

Start away from the computer & deal with the theory, but having done $x^n$, don't be afraid to try $y=x^n$ in the computer with $f'(x)=nx^{n-1}$. (You'll have to specify n, so play safe at first with n=2). The range $x=-2$ to $x=2$, $y=-2$ to $y=2$ is fairly safe. Try the *chord* option to see if the gradient of the chord tends to the value of the derivative...

An option to change the constant will appear, so try $n=-1$, $n=1/2$, as you wish. Note that if $n=1/2$ then $x^n$ is not defined for $x$ negative. Does the formula cope for $x$ positive ?

(Incidentally, for $n=-1$ the gradient option might hiccup if a chord is drawn from one side of the origin to the other. This is not a "bug". It is a genuine mathematical phenomenon. If you try $x=-10$ to $x=10$, $y=-10$ to $y=10$, then use the gradient option for $c=1$, you'll see what I mean. For any *fixed* value of $c$ there will always be values of $x$ with $x<0$ but $x+c>0$. This is why no *fixed* value of $c$ will work and why we need to take *limits*. This is a good point to discuss the need for limits.)

p.107 Exercise 5c: Allow the students to type in their own formulae again. Remember $\sqrt{}$ is typed as sqr (for square root). All the others are fairly straightforward (e.g. (4) is $x^{1/3}$ (6) is $x^{-1/2}$) Only (12) and (16) will give trouble as the computer does not have cube roots and fourth roots. (Try $x^{1/4}$ and $x^{2/3}$.) Note also that the computer sometimes cannot cope with fractional powers of a negative number. (e.g. $x^{1/3}$ is not defined for negative $x$, though mathematically it is OK. As a real fit of virtuosity you might try sgnx(absx)$^{1/3}$, and even that is not defined at the origin...)

As the "rules" come on pp.105-109, demonstrate as you feel, But once again, let students type in their functions and draw gradients + derivatives for selected examples from 1-12. Exs 13-24 can be tested by typing in $f(x)$ & $f'(x)$ and using the chord option. Even exs.25-36 can be investigated using the computer. Play it by ear, but keep a record of what happens.

p.110 TANGENTS & NORMALS

I would like the experimental group to use SUPERZOOM to attack tangents empirically. The program draws graphs. Note that the picture is not square. If you wish tangents & normals to be at right-angles, use options C (graph thro' centre) or M (input mid y-value) to get equal scales on the x and y axes....

[1] Use SUPERZOOM for teacher demonstration & class discussion. Take option G and start with the example

$$y=x^2-3x+2$$

from $x=-1$ to $x=3$, $y=-1$ to $y=3$. When drawn, select option Z for the zoom option, to be faced with a mind-boggling choice of alternatives. Touch G to keep the screen-cursor on the graph as the keyboard cursor moves it left or right. Move left or right and touch T to draw a tangent wherever you fancy. How does the computer do it? (Touching T again will remove the tangent, so you don't have to permanently mess up your picture.)

See if the students can come up with a technique. (The method is to draw the line through the point $(x,f(x))$, $(x+k,f(x+k))$ where $k=1E-5 = 0.00001$.)

Play about with option L. One press fixes the current cursor position, then (after a move somewhere else) the second press of L draws the line through the two points.

Use option L to draw a line through two points on the graph, say from $x=1$ to $x=1.001$. The technique is this:

(a) touch G to make sure the cursor is on the graph
(b) if necessary, touch X to input $x=0$
(c) press L to set this point as the first point on the line
(d) touch X to input $x=.001$
(e) touch L to set this as the second point on the line.

Does this give a fair approximation to the tangent?

Does it always work?

[6] Draw the graph $f(x)=abs(x^2-1)$ over the range $x=-2$ to 2, $y=-1$ to 3 and try drawing the tangent at $x=-1$ or $x=1$. What happens? What happens if the line is drawn from $x=1$ on the graph to $x=1-1/1000$ ?

Move the cursor on the graph to x=1 and use option I to zoom into the curve. Touch R to redraw to a higher magnification. Show that here the graph magnifies to two line segments at an angle. The method of drawing a numerical approximation only works when the graph magnifies to a straight line. In this case emphasize that the graph is not considered to have a mathematical tangent.

[7] Emphasize the case that when we can calculate a derivative at the point then the graph has a tangent there with gradient of tangent equal to the value of the derivative.

(Students who aren't shown this explicitly are liable to interpret the ideas "gradient", "derivative" and "tangent" in slightly differing contexts, e.g. a graph can have no gradient at a point (because it doesn't magnify to a straight line) but *can* have a tangent, which is a line that *touches* the graph. According to their interpretation, abs($x^2$-1) has no gradient at x=-1 or x=1, but does have a tangent, namely the x-axis. Have an informal chat with the students to see if they can cope with the following ideas:

Mathematically we say the graph of abs($x^2$-1) doesn't have a gradient at x=1 or x=-1, though you could say it has a gradient to the left & a gradient to the right and a tangent to the left & a tangent to the right, which happen to be different. In A-level we shall only be concerned with cases where the left and right tangents are the same, cases where the graph magnifies to a straight line, where the gradient to the left and right are the same and give the gradient of the unique tangent at the point. I would be interested to see if they find this concept easy or hard.)

Can the students suggest other graphs which don't have tangents at certain points?

In examples 5e you might try to draw the curve over an appropriate range, then superimpose the straight line graph given by the tangent formula.

E.g. Draw $y-x^2-3x+2$ over the ranges $x=-2$ to 2, $y=-1$ to 3 and then touch G to draw a graph again, input $y=-3x+2$ and superimpose it.

p.112 Exercise 5e: Allow the students to draw some of the curves using SUPERZOOM and superimpose the tangent (by drawing the graph using the tangent formula...)

At this stage I should like the students to do the investigation "THE IDEA OF A TANGENT" using the computer. (See note on administration of tests.)

***********

p.113 STATIONARY VALUES

[1] The computer can be used with the GRADIENT program to draw $f(x)=x^3-3x^2+2$ over the range x=-4 to 4, y=-4 to 4. Option G, the gradient function, for c=1/100 approximates to the derivative and shows the derivative to be zero at the maximum and minimum.

[2] Again, allow students to draw some of the graphs in Ex.5f, using the system specified before.

If you consider it appropriate, show a function such as $f(x)=abs(x^2-1)$ to show that it has minimum values at -1 and +1, but no derivative here. Emphasise the rule that *if* there is a maximum or minimum and *if* the function has a derivative, then the derivative must be zero. But maxima and minima can also occur where the functions have no derivative.

P.114, TURNING POINTS. [1] Use the gradient function option in GRADIENT to show the properties of maxima and minima dynamically. Try $f(x)=x^3-x$ over the range x=-2 to 2, option C for graph through centre screen. Draw the gradient function for c=1/100 and see how the gradient is positive just before a maximum and negative just after, and vice-versa for a minimum. You can use the options S, space to slow down or stop during your demonstration, or touch a number to plot more points to slow it down. Get the students to suggest for what values of x there is a maximum or minimum. They will need to do a calculation to do it!

(The derivative is zero at the maximum, so $f'(x)=0$, giving $3x^2=1$ and $x= 1/ 3$. You can use the chord option C to draw the chord through a=-1/sqr3, b=0 and let b tend to a, to see if numerically the chord gradient tends to zero. The value will eventually be displayed as -0.0000, indicating a very small negative number. You could also try other starting values, e.g. a=-1/sqr3, b=-1 or a=1/sqr3, b=0, etc.)

p.116 It is worth emphasizing that, having found $f'(a)=0$, then simply calculating numerical values of $f(x)$ for $x$ on either side of a, is a good way to check whether a is a max or min. When $f'(x)$ exists throughout and *all* the zeros of $f'(x)$ are known this is an absolutely watertight method of finding maxima and minima. After students have met the rule about the second derivative, they usually go blindly for that and often involve themselves in a lot of algebraic manipulation to calculate $f''(a)$ to prove something that they could see easily with a couple of simple numerical calculations.

p.117 & 118 INVESTIGATING THE NATURE OF STATIONARY VALUES

GRADIENT does not draw the second derivative. (If it did, the picture could get quite complicated!) But the gradient function option to draw an approximation to the first derivative can still help with ideas about the second derivative, because the second derivative is simply the gradient function of the $f'(x)$.

[1] Use GRADIENT to draw $f(x)=x^3-x$, (x-range -2 to 2, option C for graph through the centre.) Select option G and use c=1/100 to approximate the derivative $f'(x)=1-3x^2$ by the gradient function. Get the students to look at the gradient function as it is being drawn. Slow it down by touching S and introduce more points by touching 5. Note that at the local maximum of $f(x)$, $f'(x)$ is positive just before and negative just after. Concentrate on the picture of the gradient function. (The options include one to change the display to show the gradient function only.) Regarding the gradient function as a fair approximation for the derivative $f'(x)$, we see that $f'(x)$ has negative gradient so its derivative, the second derivative $(f')'(x)$, is negative at the maximum. The situation is reversed at the minimum, where $f'(x)$ is negative just before, positive just after and the second derivative is positive.

In this case we get:

| Maximum | Minimum |
|---|---|
| $f'(x)=0$ | $f'(x)=0$ |
| $f''(x)<0$ | $f''(x)>0$. |

Is this always so?

Look at the example $f(x)=x^4+1$ (over the range x=-2 to 2, y=-1 to 3), and draw the gradient function for c=1/100 to give an approximation to the derivative $f'(x)=4x^3$. At the origin $f(x)$ has a minimum, $f'(x)=0$ and the derivative is negative before zero and positive after, but momentarily at zero its gradient $f''(x)$ is zero. Thus at a maximum $f''(x)$ may be also be zero.

Draw the graph of $f(x)=x^4-1$ over the same range (if you touch RETURN when the range is requested, it keeps the same range). This gives a picture of a *maximum* where $f'(x)=0$, $f''(x)=0$.

The case $f(x)=x^3+1$ over the same range has a horizontal inflexion (neither maximum nor minimum) where $f'(x)=0$, $f''(x)=0$.

Thus the full picture is

| Maximum | Minimum | Horizontal Inflexion |
|---|---|---|
| $f'(x)=0$ | $f'(x)=0$ | $f'(x)=0$ |
| $f''(x)\ 0$ | $f''(x)\ 0$ | $f''(x)=0$. |

These ideas are deeper than others mentioned earlier. The computer program can help if the students can mentally picture the gradient option G to draw the derivative of a given graph, so this depends partly on the success of the earlier demonstrations. Let me know how successful, or unsuccessful this is with the members of your group.

[2] p.122 Exercises: Again students can draw the graphs with GRADIENT, use option G to draw the gradient function approximation to the derivative and check whether they have the

correct answers for maxima and minima. As they draw the graphs they should look at the sign of $f'(x)$ near each maximum or minimum $x=a$, and at their calculated value of $f''(a)$.

After completing the work in the chapter, all students should take the post-test. (INVESTIGATIONS INTO IDEAS OF THE CALCULUS). This is eight pages long and has the same first three pages as the pre-test, so don't get the two mixed up...

## Appendix 2

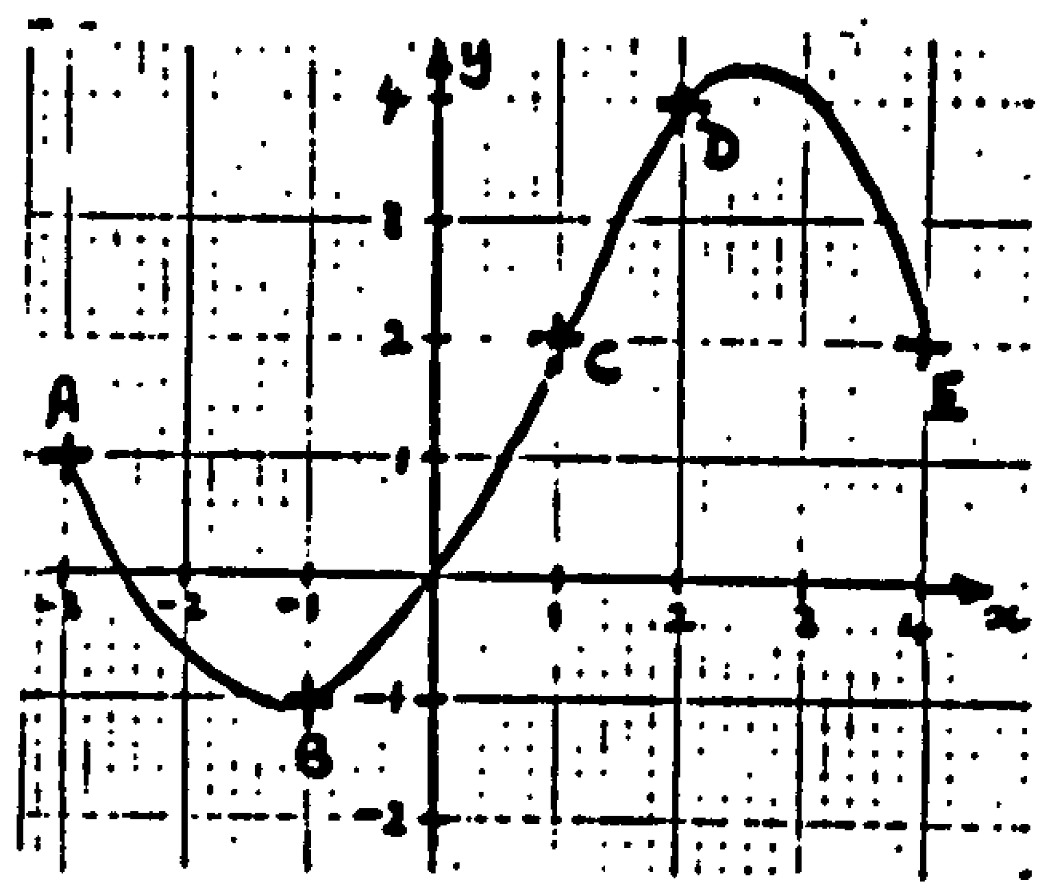
### Pre-test

INVESTIGATION INTO IDEAS OF THE THE CALCULUS **A**

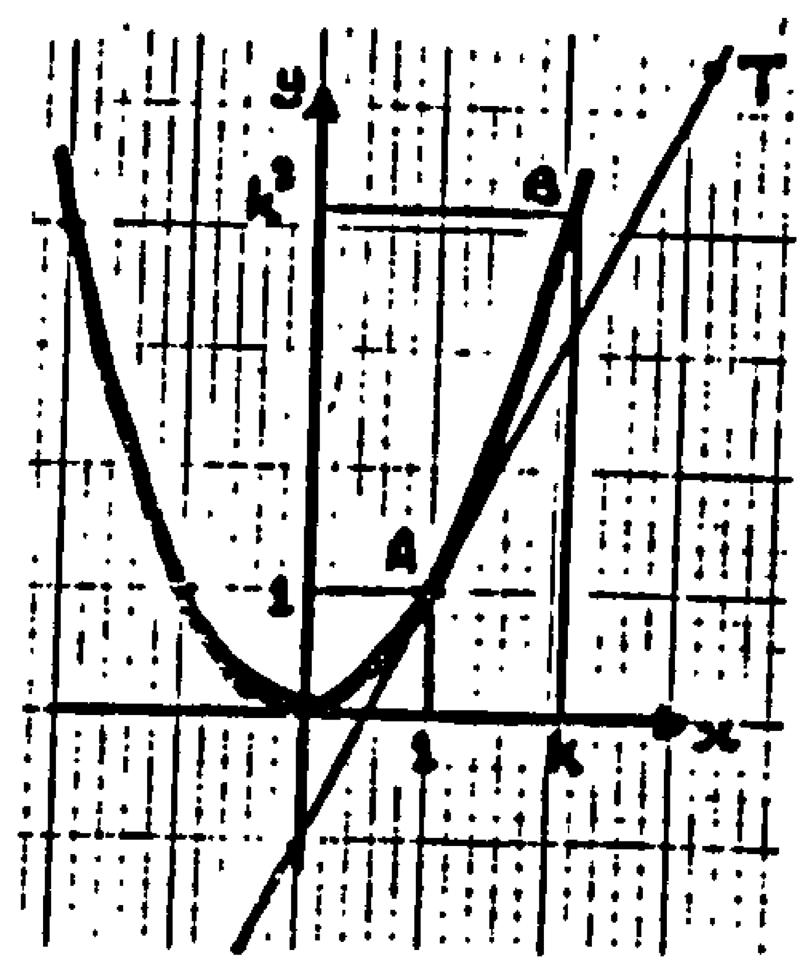
*This is not a test. It is simply an investigation to see how people view ideas in the calculus. Please answer each question as best as you can and try not to leave any question unanswered.*

1.Find the average rate of change between the following points on the graph:
*(Note: the "average rate of change" from P to Q means the gradient of PQ)*

(i) from C to D .....
(ii) from D to E .....
(iii) from A to B .....
(iv) from B to C .....
(v) from C to E .....
(vi) from D to C .....




2.

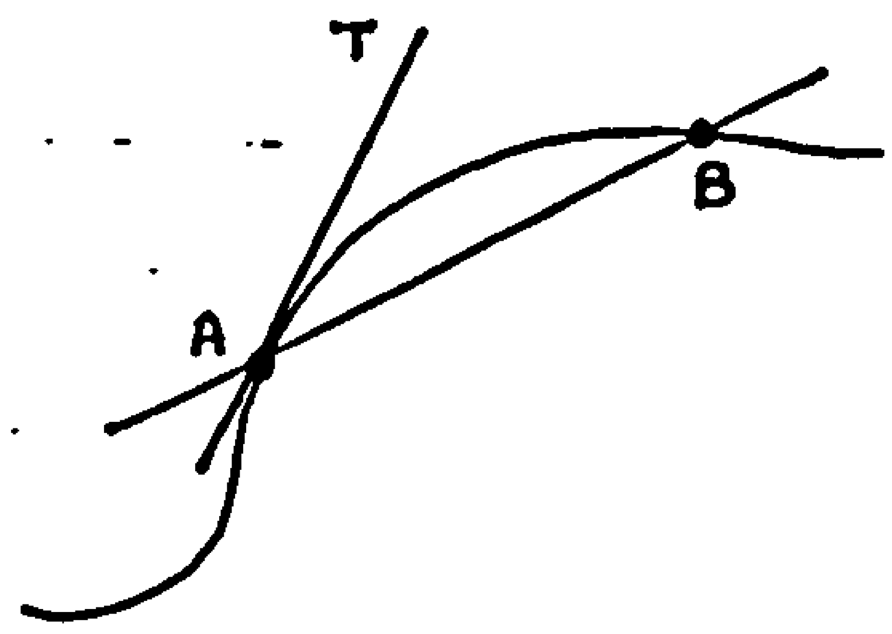



On the graph $y=x^2$, the point A is (1,1), the point B is $(k,k^2)$ and T is a point on the tangent to the graph at A.

(i) Write down the gradient of the straight line through A,B....

(ii) Write down the gradient of AT.....

Explain how you might find the gradient of AT from first principles.




3. Referring to the diagram, which of the following statements would you say are true and which are false. Read each statement very carefully. If you are ABSOLUTELY SURE, underline the response in CAPITALS, otherwise underline the response in lower case letters.
In each statement the "line through two points" or the "tangent" means the whole line, not just the line segment between the two points concerned.

(a) as B→A, the line through AB tends to the tangent AT.
TRUE/true/false/FALSE (underline one response).

(b) As B→A, the line through AB has the tangent AT as a limit.
TRUE/true/false/FALSE (underline one response).

(c) As B→A the line through AB reaches the tangent as a limit.
TRUE/true/false/FALSE (underline one response).

(d) As B→A the line through AB reaches the tangent in the limit.
TRUE/true/false/FALSE (underline one response).

(e) As B→A the line through AB approaches the tangent as a limit.
TRUE/true/false/FALSE (underline one response).

(f) As B→A, the line through AB becomes equal to the tangent at T.
TRUE/true/false/FALSE (underline one response).

(g) As B→A the line through AB becomes practically indistinguishable from the tangent at T.
TRUE/true/false/FALSE (underline one response).

(h) The tangent AT is the line through two very close points on the graph.
TRUE/true/false/FALSE (underline one response).

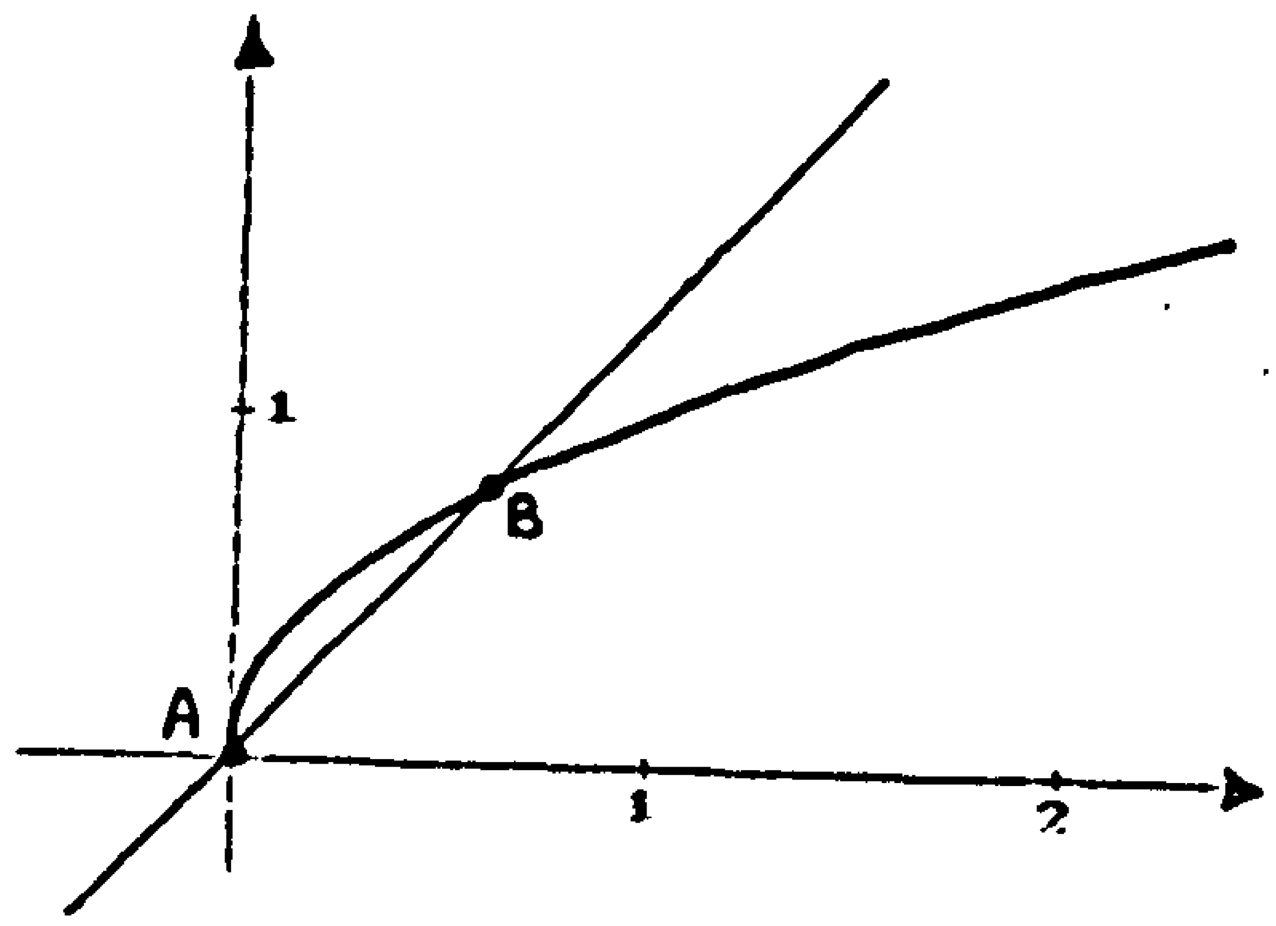
(i) The tangent AT is the line through two coincident points on the graph at A.
TRUE/true/false/FALSE (underline one response)

(j) $\lim_{B \to A}$ (gradient of chord AB) = gradient of tangent AT.
TRUE/true/false/FALSE (underline one response)

(k) as B tends to A, the gradient of the chord AB tends to the gradient of the tangent.
TRUE/true/false/FALSE (underline one response)

(l) as B→A the limit of the gradient of the chord AB is the gradient of the tangent AT.
TRUE/true/false/FALSE (underline one response)

4. The diagram represents the graph of the function $y=\sqrt{x}$ (taking the positive square root for $x \geq 0$). A is the point (0,0) and B is the point $(h, \sqrt{h})$. Using the same conventions as question 3, underline the appropriate response for each of the following statements:




(a) The graph has a tangent at A.
TRUE/true/false/FALSE (underline one response)

If your response is "false" (or FALSE!), explain why in the following space, then omit (b) and (c)...

(b) The tangent at A is vertical.
TRUE/true/false/FALSE (underline one response)

(c) The gradient of the tangent is infinite.
TRUE/true/false/FALSE (underline one response)

(d) As $B \to A$, the gradient of the line AB tends to infinity.
TRUE/true/false/FALSE (underline one response)

(e) As $B \to A$, the gradient of the line AB has infinity as its limit.
TRUE/true/false/FALSE (underline one response)

(f) As $B \to A$, the gradient increases without limit.
TRUE/true/false/FALSE (underline one response)

5. If you have studied the calculus before, write down the derivatives of the following:

(a) $x^4+3x^2$ ......

(b) $\sqrt{x}$ .....

(c) $1/x^2$ .....

Please write your name ..............

*Thanks for your help!*

## Appendix 3

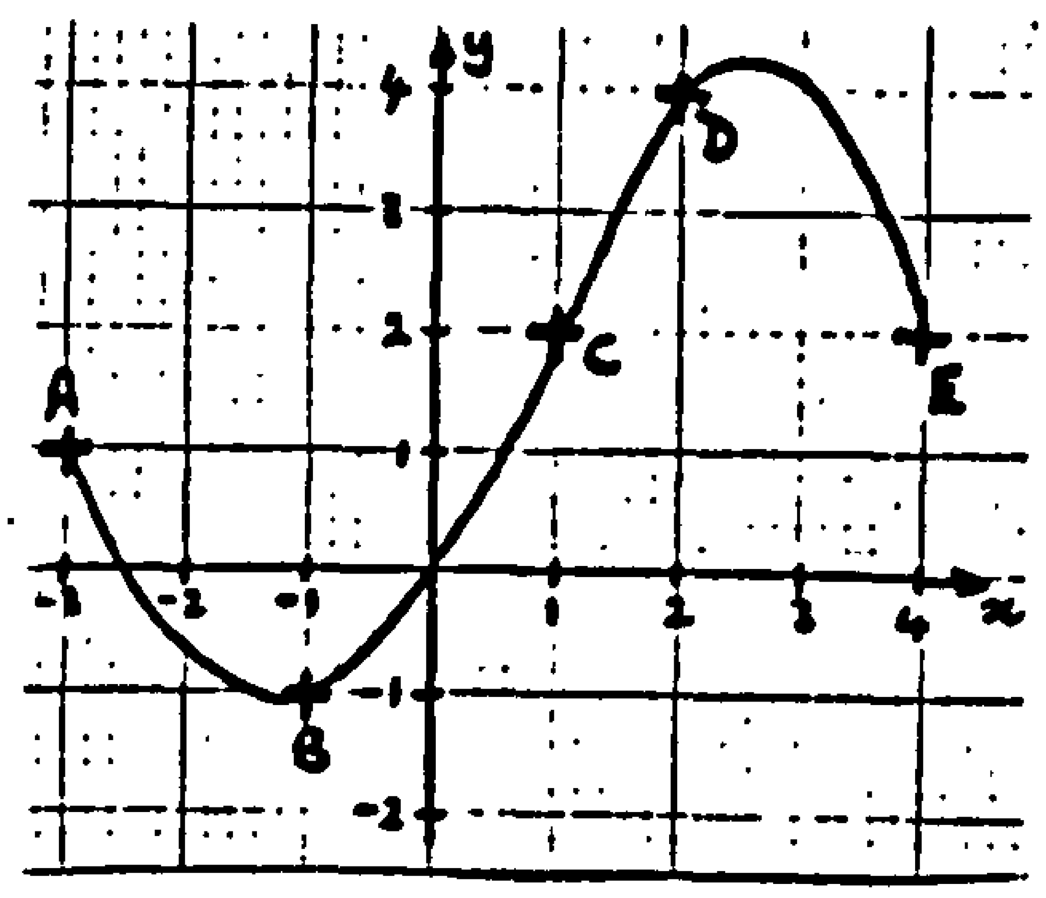
## Post-test

INVESTIGATION INTO IDEAS OF THE THE CALCULUS **B**

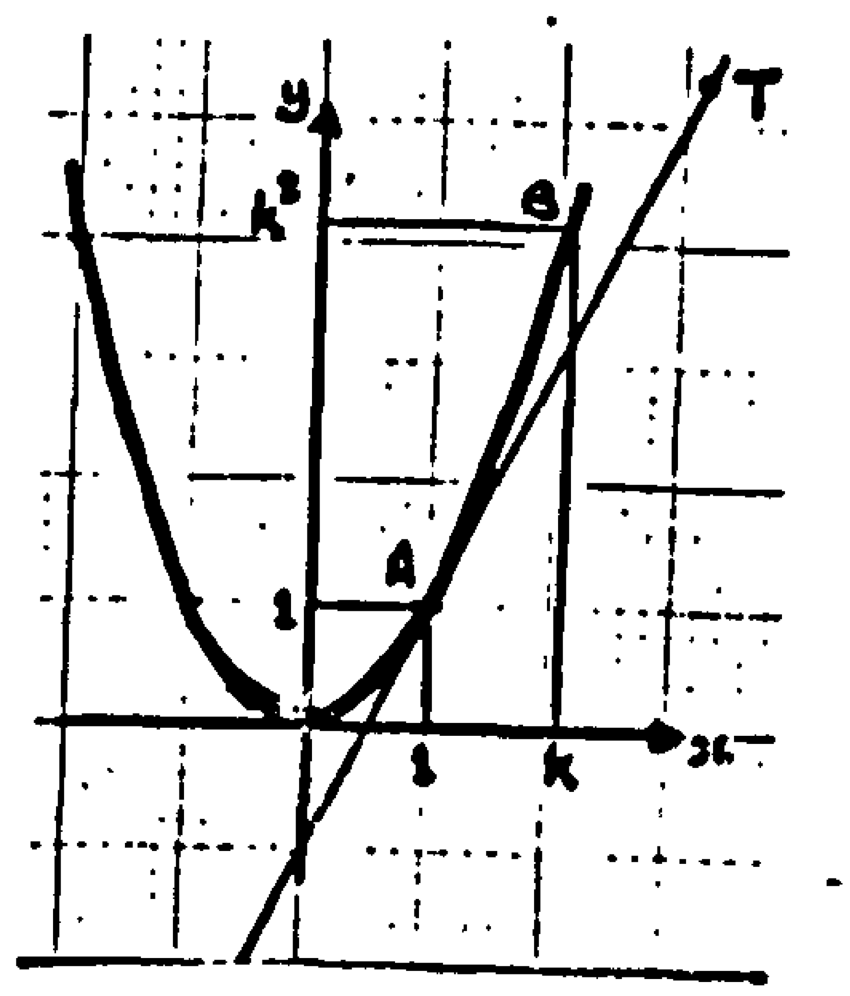
*This is not a test. It is simply an investigation to see how people view ideas in the calculus. Please answer each question as best as you can and try not to leave any question unanswered.*

1. Find the average rate of change between the following points on the graph:
*(Note: the "average rate of change" from P to Q means the gradient of PQ)*

(i) from C to D .....
(ii) from D to E .....
(iii) from A to B .....
(iv) from B to C .....
(v) from C to E .....
(vi) from D to C .....




2.



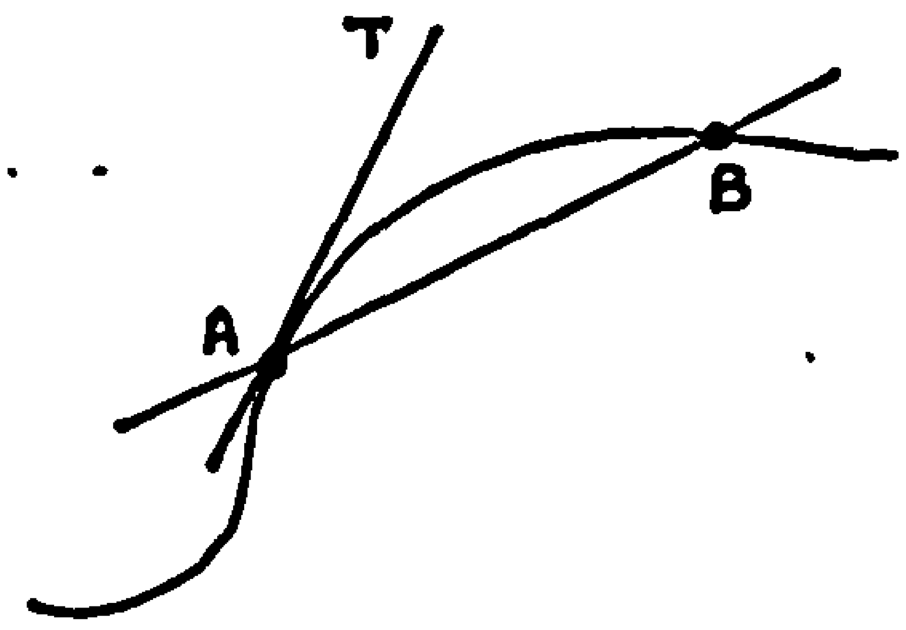

On the graph $y=x^2$, the point A is (1,1), the point B is $(k,k^2)$ and T is a point on the tangent to the graph at A.

(i) Write down the gradient of the straight line through A,B....

(ii) Write down the gradient of AT.....

Explain how you might find the gradient of AT from first principles.

3. Referring to the diagram, which of the following statements would you say are true and which are false. Read each statement very carefully. If you are ABSOLUTELY SURE, underline the response in CAPITALS, otherwise underline the response in lower case letters.
In each statement the "line through two points" or the "tangent" means the whole line, not just the line segment between the two points concerned.




(a) as $B \to A$, the line through AB tends to the tangent AT.
TRUE/true/false/FALSE (underline one response).

(b) As $B \to A$, the line through AB has the tangent AT as a limit.
TRUE/true/false/FALSE (underline one response).

(c) As $B \to A$ the line through AB reaches the tangent as a limit.
TRUE/true/false/FALSE (underline one response).

(d) As $B \to A$ the line through AB reaches the tangent in the limit.
TRUE/true/false/FALSE (underline one response).

(e) As $B \to A$ the line through AB approaches the tangent as a limit.
TRUE/true/false/FALSE (underline one response).

(f) As $B \to A$, the line through AB becomes equal to the tangent at T.
TRUE/true/false/FALSE (underline one response).

(g) As $B \to A$ the line through AB becomes practically indistinguishable from the tangent at T.
TRUE/true/false/FALSE (underline one response).

(h) The tangent AT is the line through two very close points on the graph.
TRUE/true/false/FALSE (underline one response).

(i) The tangent AT is the line through two coincident points on the graph at A.
TRUE/true/false/FALSE (underline one response)

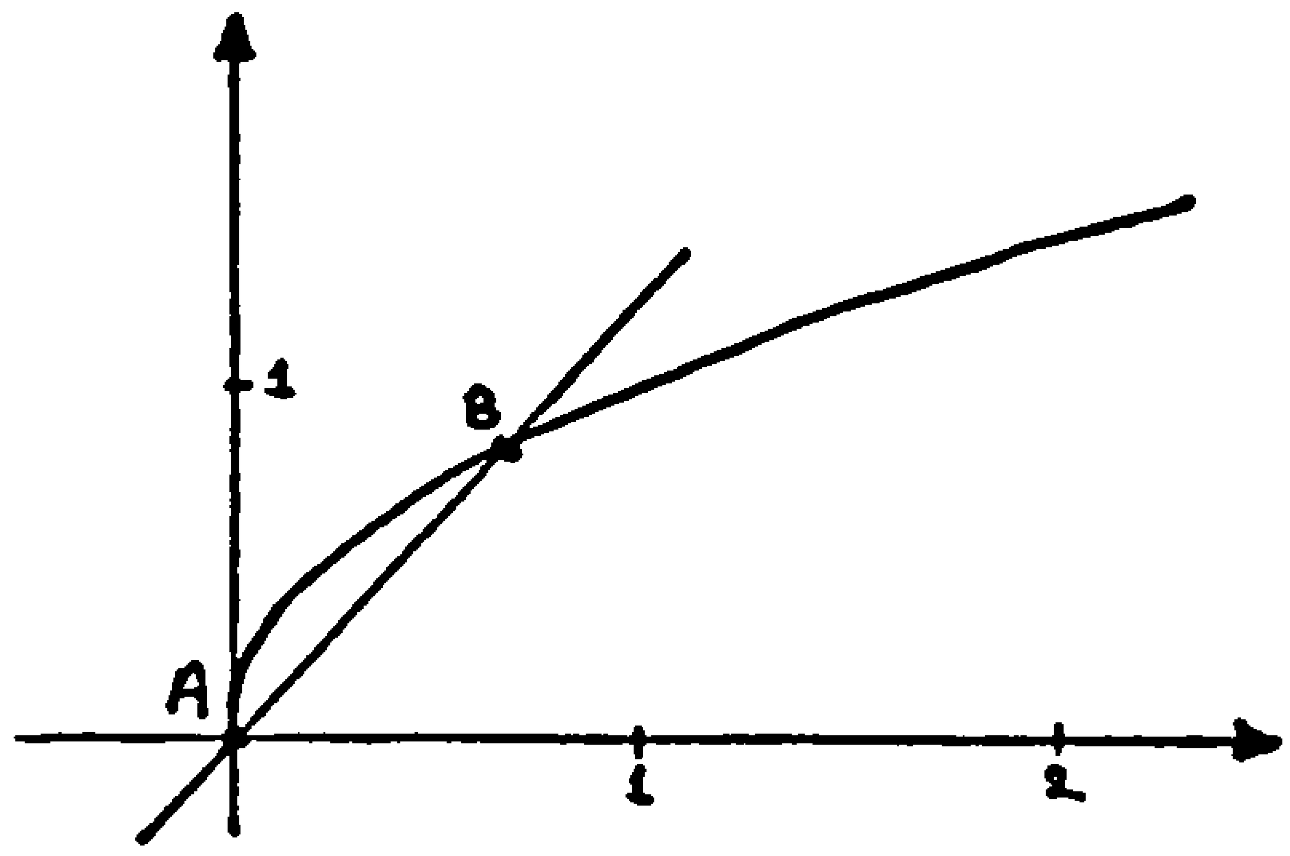
(j) $\lim_{B \to A}$ (gradient of chord AB) = gradient of tangent AT.

TRUE/true/false/FALSE (underline one response)

(k) as B tends to A, the gradient of the chord AB tends to the gradient of the tangent.
TRUE/true/false/FALSE (underline one response)

(l) as $B \to A$ the limit of the gradient of the chord AB is the gradient of the tangent AT.
TRUE/true/false/FALSE (underline one response)

4. The diagram represents the graph of the function $y=\sqrt{x}$ (taking the positive square root for $x\geq 0$). A is the point (0,0) and B is the point $(h,\sqrt{h})$. Using the same conventions as question 3, underline the appropriate response for each of the following statements:




(a) The graph has a tangent at A.
TRUE/true/false/FALSE (underline one response)

If your response is "false" (or FALSE!), explain why in the following space, then omit (b) and (c)...

(b) The tangent at A is vertical.
TRUE/true/false/FALSE (underline one response)

(c) The gradient of the tangent is infinite.
TRUE/true/false/FALSE (underline one response)

(d) As $B\to A$, the gradient of the line AB tends to infinity.
TRUE/true/false/FALSE (underline one response)

(e) As $B\to A$, the gradient of the line AB has infinity as its limit.
TRUE/true/false/FALSE (underline one response)

(f) As $B\to A$, the gradient increases without limit.
TRUE/true/false/FALSE (underline one response)

5. Write down the derivatives of the following:

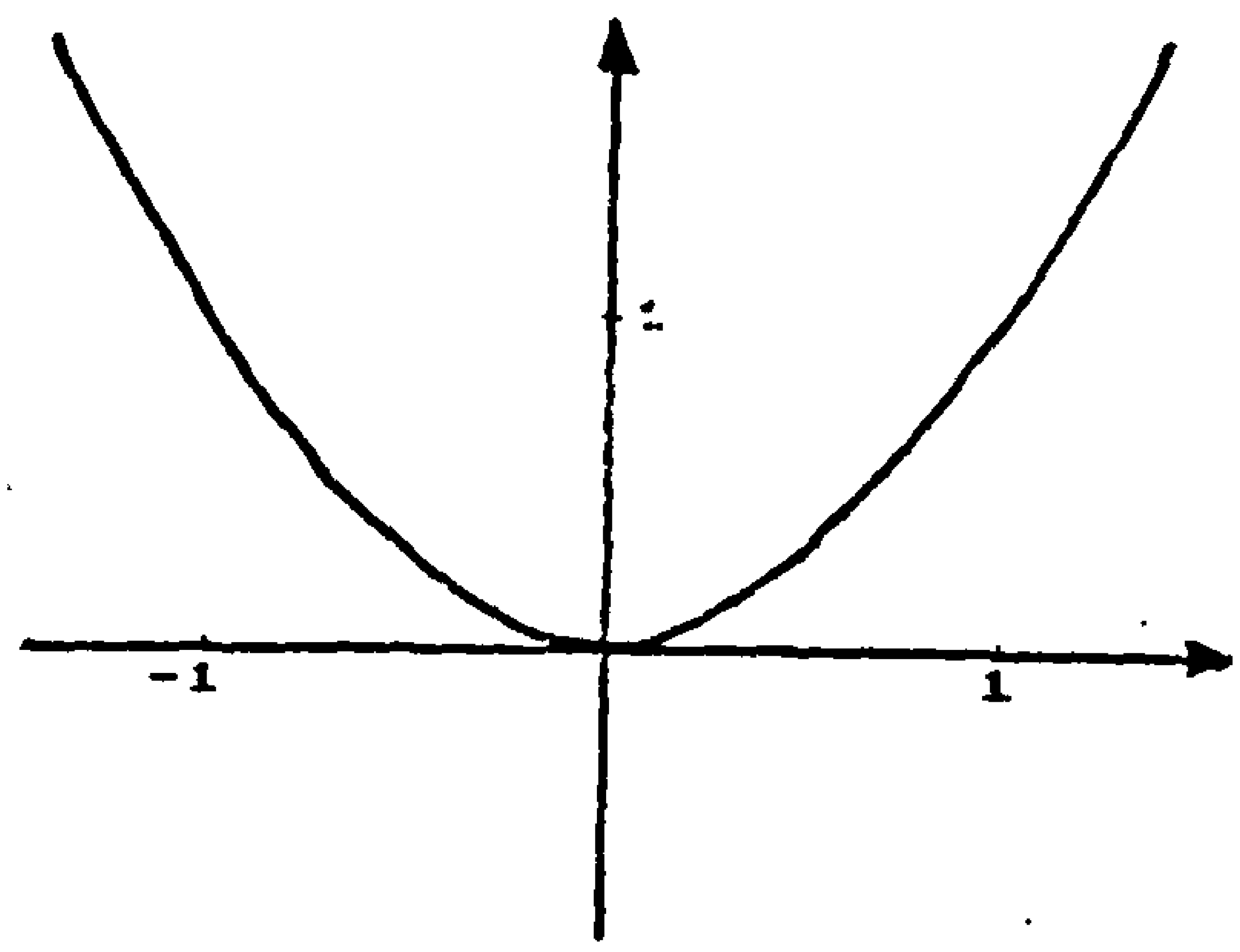
(a) $x^4+3x^2$ ......

(b) $\sqrt{x}$ ......

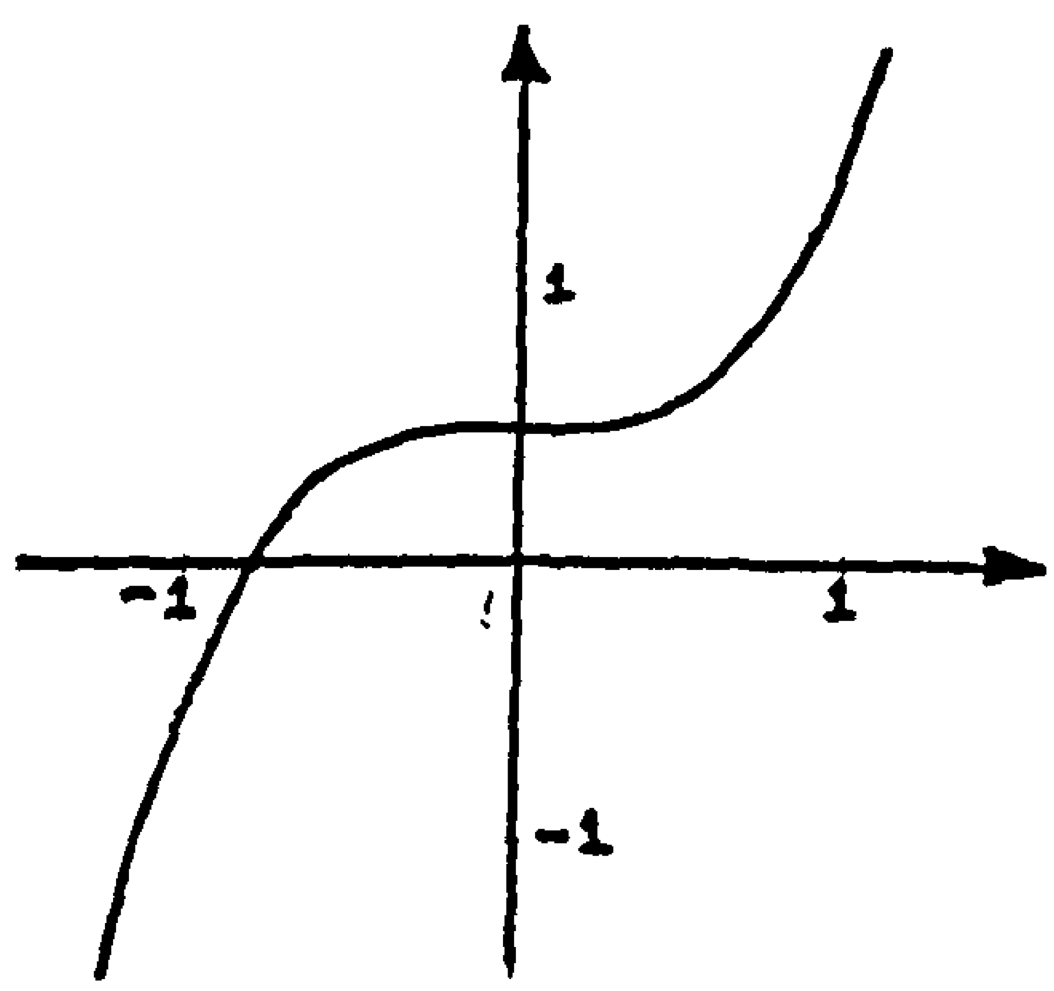
(c) $1/x^2$ .....

6. Sketch the *derivatives* of the following graphs:

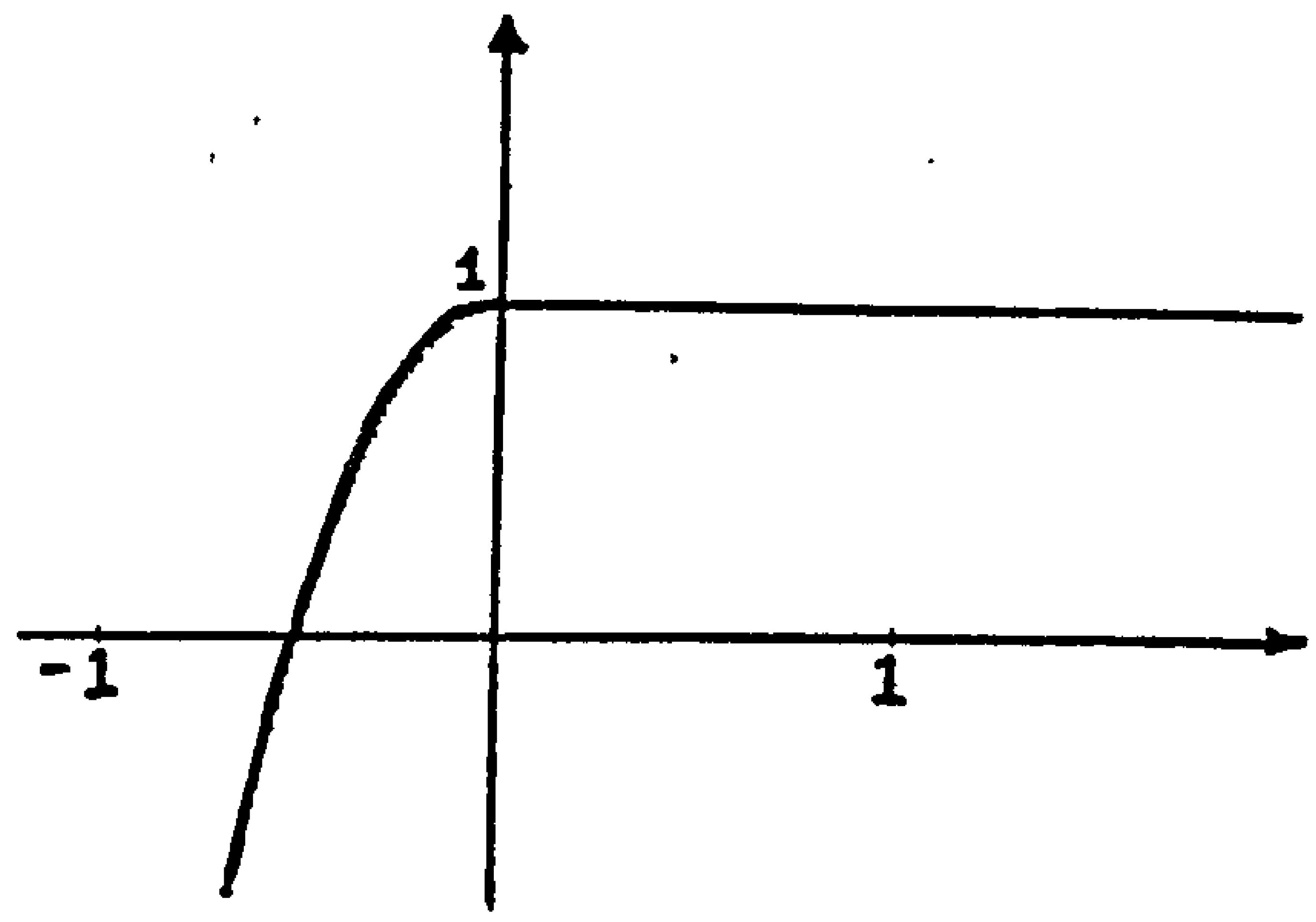
(a)



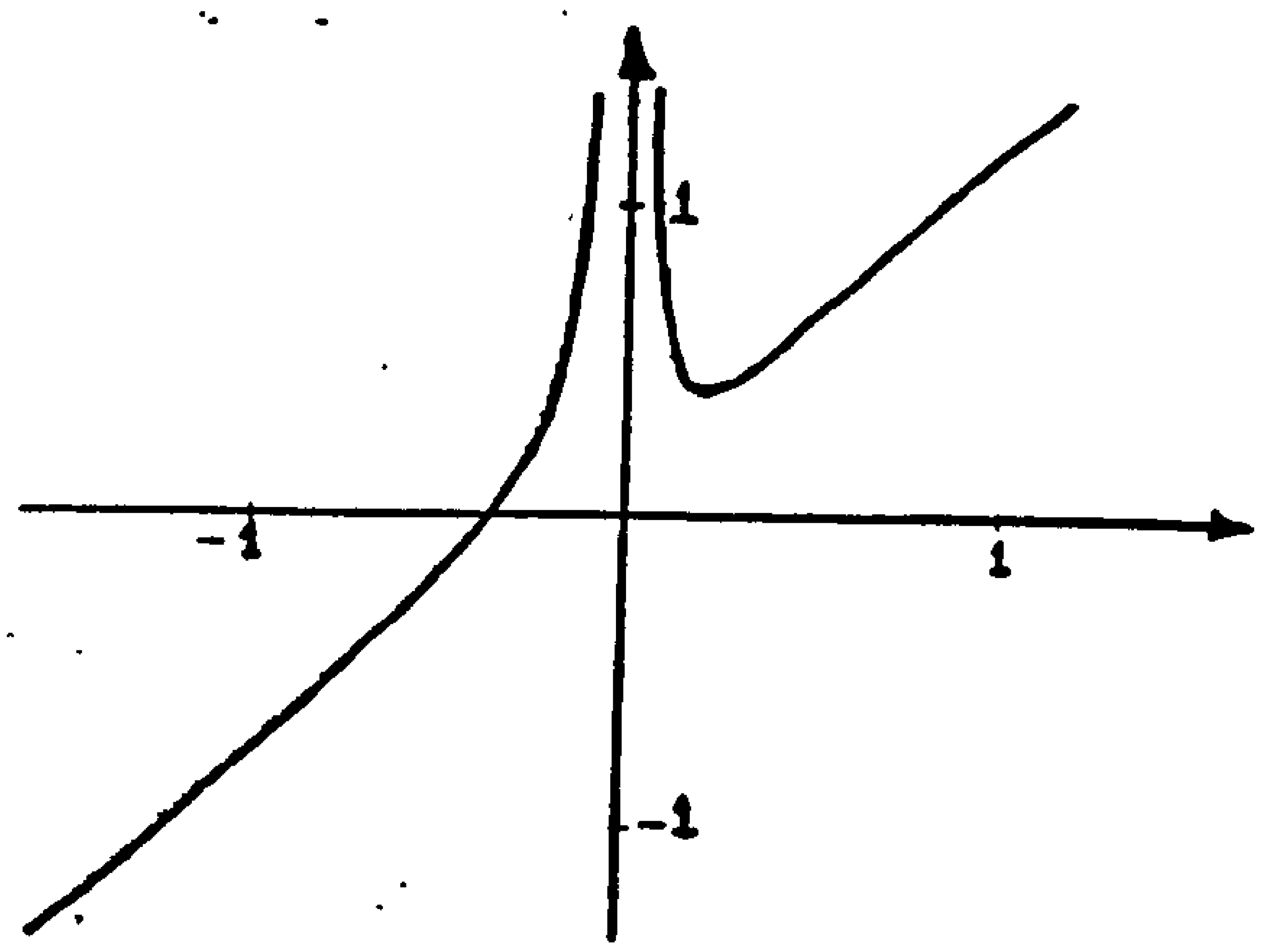

(b)



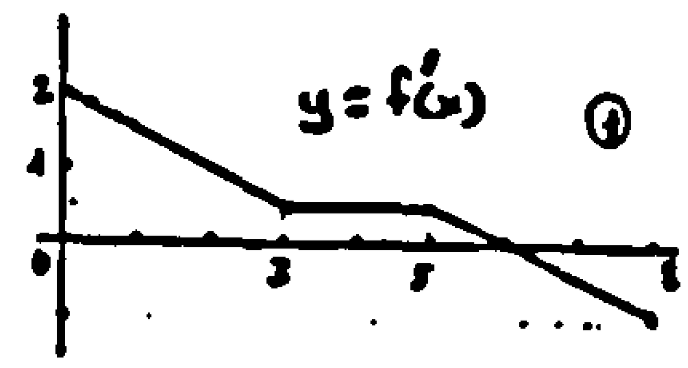
(c)




(d)

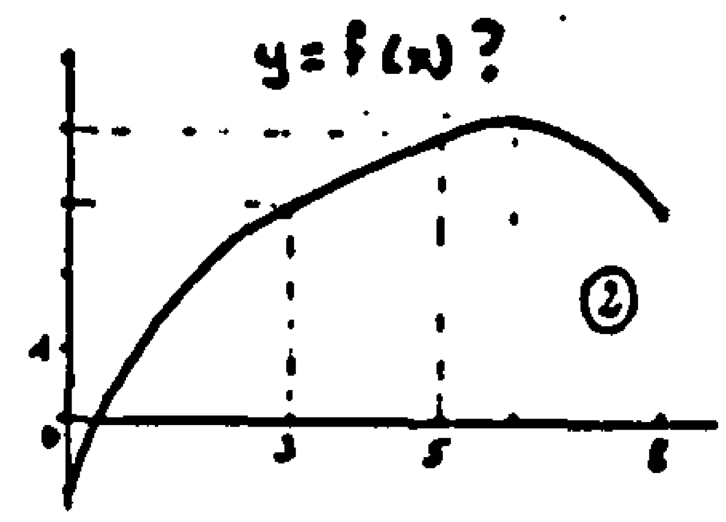


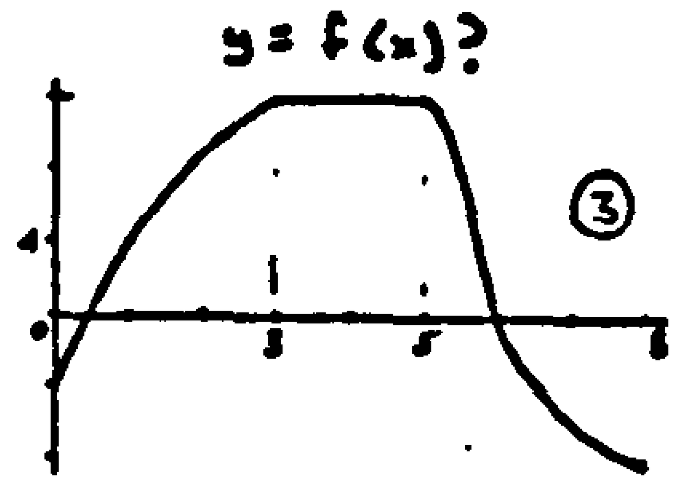
7. Graph 1 is the *derivative* y=f'(x) of a function y=f(x) defined for 0≤x≤8.

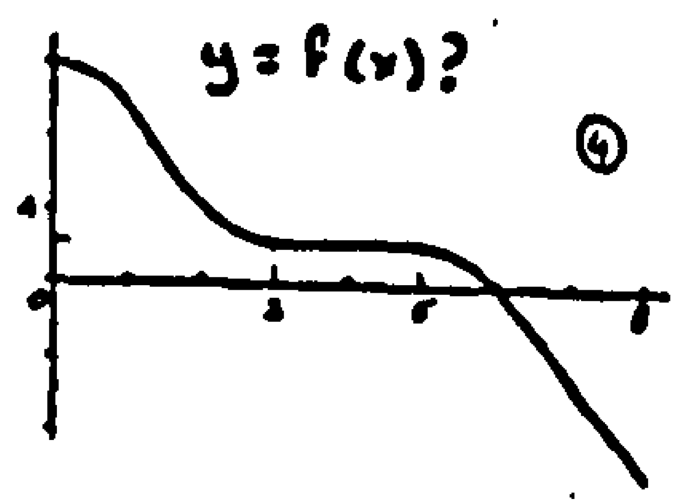



Which of the graphs 2,3,4 could be the *original* graph y=f(x)?

Give the reason(s) for your choice:




8. Give an example of a function which is defined at x=1 but is not differentiable at x=1.

9. You have been asked by a student who understands the notion of the gradient of a straight line to explain what is meant by the gradient of a more general graph. Give a brief explanation:

10. Say what is meant by a tangent to a graph:

11. Say what is meant by the following symbols:

(i) $\delta x$ ...

(ii) $\delta y$ ...

(iii) $\frac{\delta y}{\delta x}$ ...

(iv) $dx$ ...

(v) $dy$ ...

(vi) $\frac{dy}{dx}$ ...

12. Explain what is meant by the *derivative* of a function.

Please write your name ...............

Were you in a group using a computer to draw gradients?

Circle one reply: Yes/No

**********************************************************

If you were in a group using a computer:

1. Did you find the computer helpful?

Circle one:

Very helpful/ helpful /fairly helpful/ neutral/ fairly unhelpful/ unhelfpul/ very unhelpful

2. How many times did you use the computer yourself or as part of a group of not more than three?

Circle one: 0 times / 1 / 2 / 3 / 4 / more than 4

3. How much time did the class as a whole spend using the computer:

far too much/ too much/ about right / too little / far too little

4. How much time did you have using the computer by yourself or with a small group?:

far too much/ too much/ about right / too little / far too little

5. In what ways did the computer help? (Specify)

6. In what ways was the computer unhelpful? (Specify)

**********************************************************

*Whether you used a computer or not, thanks for your tise.*

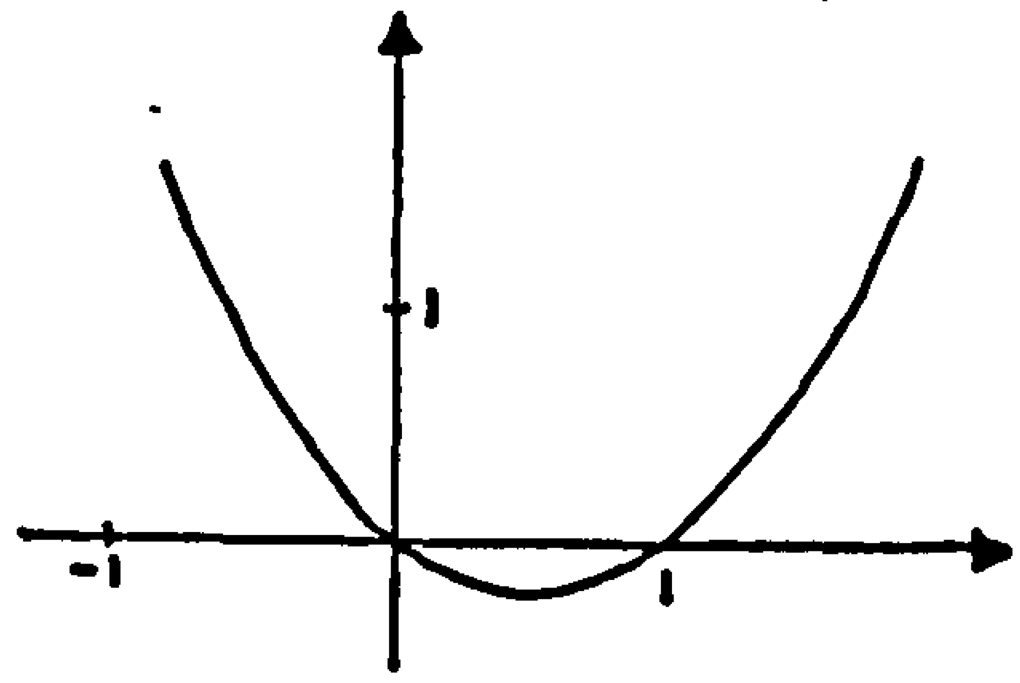
# Appendix 4

## Gradient Investigation

### *THE IDEA OF A GRADIENT*

***This is an investigation into how students think about gradients. It is not a test. Please answer the questions carefully and try to give reasons for your answers.***

**1. The graph of $y=x^2-x$**



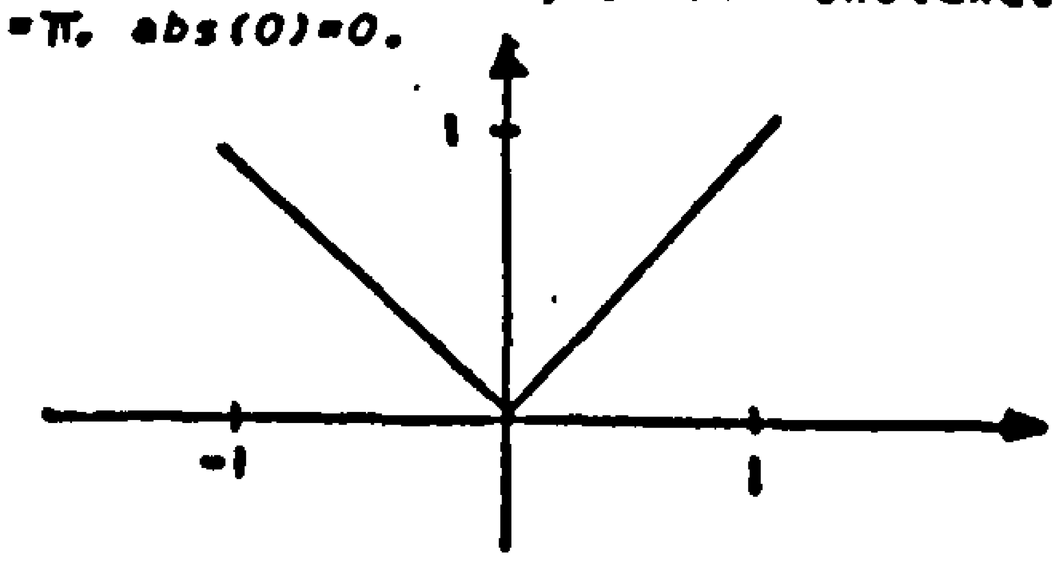

Can you calculate the gradient at x=0? YES/NO
If YES, what is the gradient? if NO, why not?

Are you sure of your answer?
Underline one response that bes fits your feelings:
Certain / fairly sure / fairly doubtful / very doubtful

*-*-*-*-*-*-*-*-*-*-*-*-*

***The absolute value of a number t is denoted by abs(t) and represents the numerical value of t, regardless of sign. Thus if t is negative, the absolute value strips off the negative sign. For instance, abs(1.5)=1.5, abs(-3.7)=3.7, abs(π)=π, abs(0)=0.***

**2. The graph of y=abs(x).**

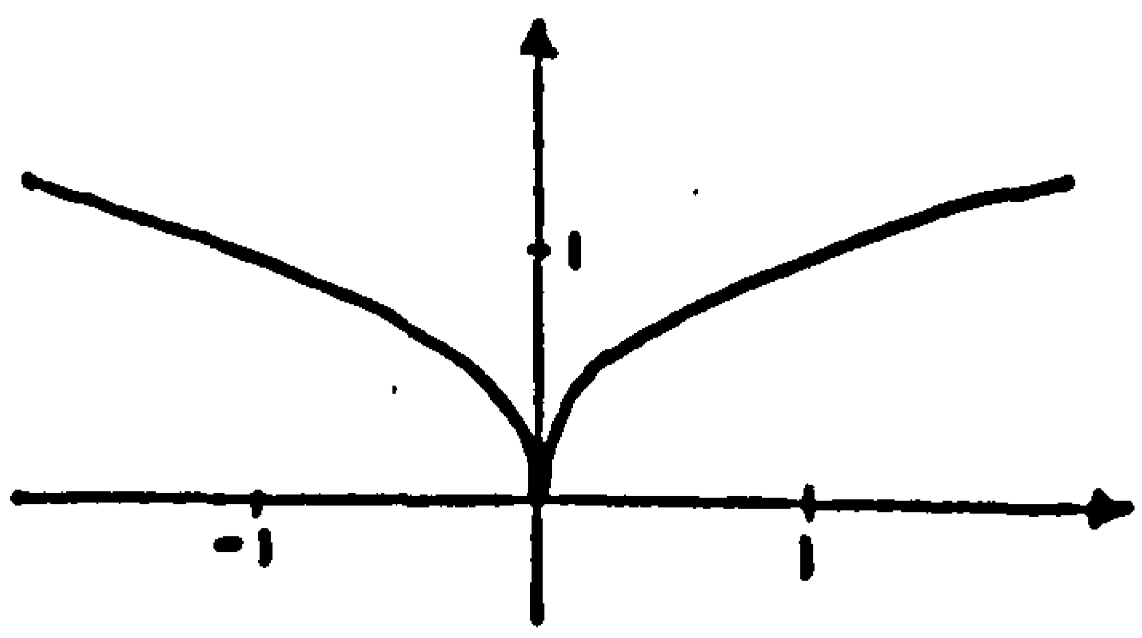



Can you calculate the gradient at x=0? YES/NO
If YES, what is the gradient? if NO, why not?

Are you sure of your answer?
Underline one response that best fits your feelings:
Certain / fairly sure / fairly doubtful / very doubtful

*-*-*-*-*-*-*-*-*-*-*-*-*

**3. The graph of y=√(abs(x))**



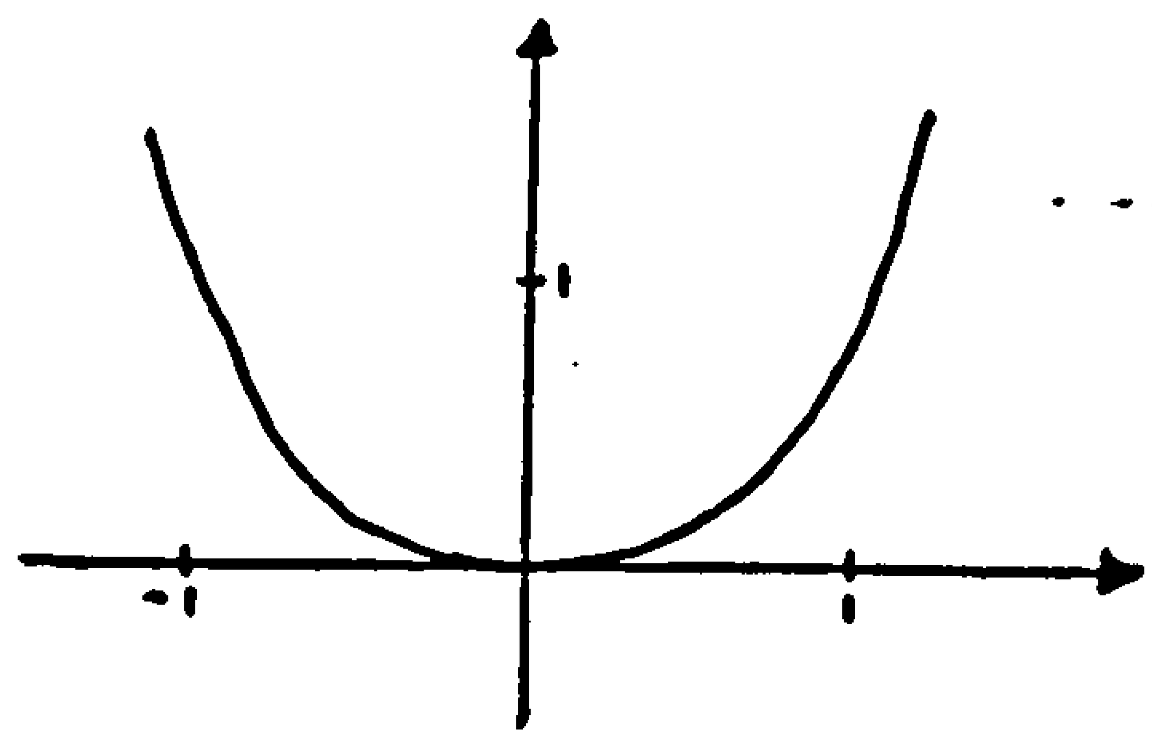

Can you calculate the gradient at x=0? YES/NO
If YES, what is the gradient? if NO, why not?

Are you sure of your answer?
Underline one response that best fits your feelings:
Certain / fairly sure / fairly doubtful / very doubtful

*-*-*-*-*-*-*-*-*-*-*-*-*

4. abs($x^3$)

Can you calculate the gradient at x=0? YES/NO
If YES, what is the gradient? if NO, why not?

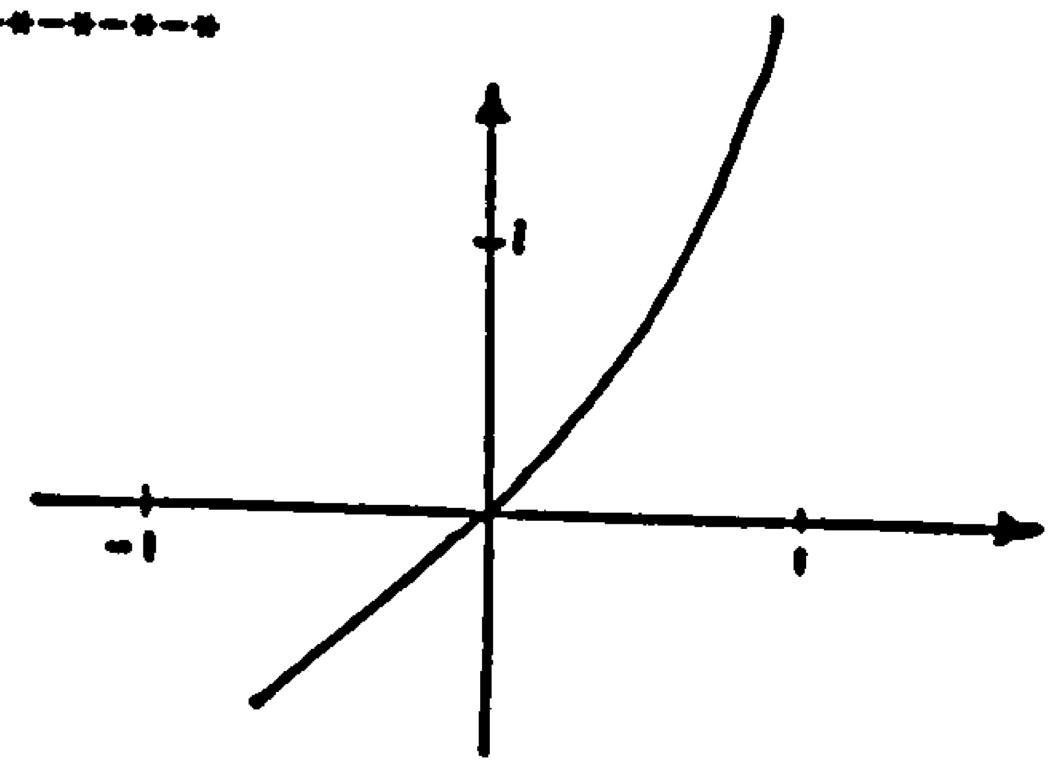



Are you sure of your answer?
Underline one response that best fits your feelings:
Certain / fairly sure / fairly doubtful / very doubtful

*-*-*-*-*-*-*-*-*-*-*-*

5. The graph of $y=\begin{cases} x & (x\leq 0) \\ x+x^2 & (x\geq 0)\end{cases}$

Can you calculate the gradient at x=0? YES/NO
If YES, what is the gradient? if NO, why not?

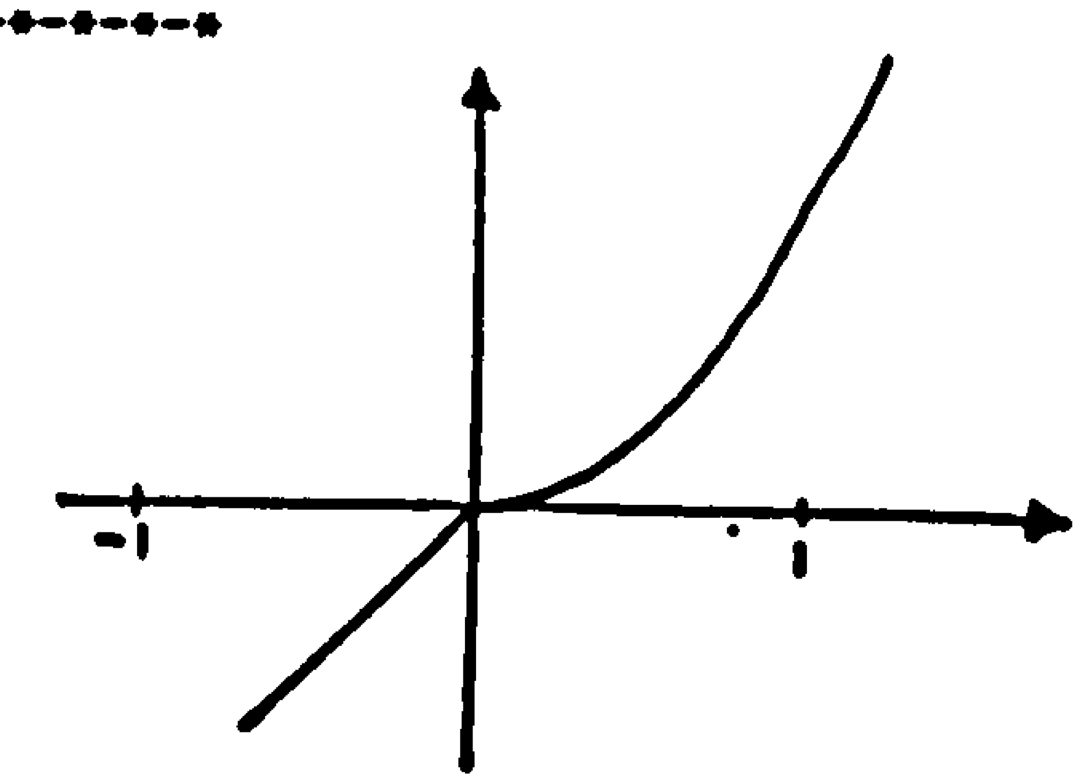



Are you sure of your answer?
Underline one response that best fits your feelings:
Certain / fairly sure / fairly doubtful / very doubtful

*-*-*-*-*-*-*-*-*-*-*-*

6. The graph of $y=\begin{cases} x & (x\leq 0) \\ x^2 & (x\geq 0)\end{cases}$

Can you calculate the gradient at x=0? YES/NO
If YES, what is the gradient? if NO, why not?




Are you sure of your answer?
Underline one response that best fits your feelings:
Certain / fairly sure / fairly doubtful / very doubtful

*-*-*-*-*-*-*-*-*-*-*-*

*Please write your name ................*

*Have you seen gradients drawn on the BBC computer? YES/NO*

*Have you used the computer to help answer these questions? YES/NO*

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*Thanks for your help...*

# Appendix 5

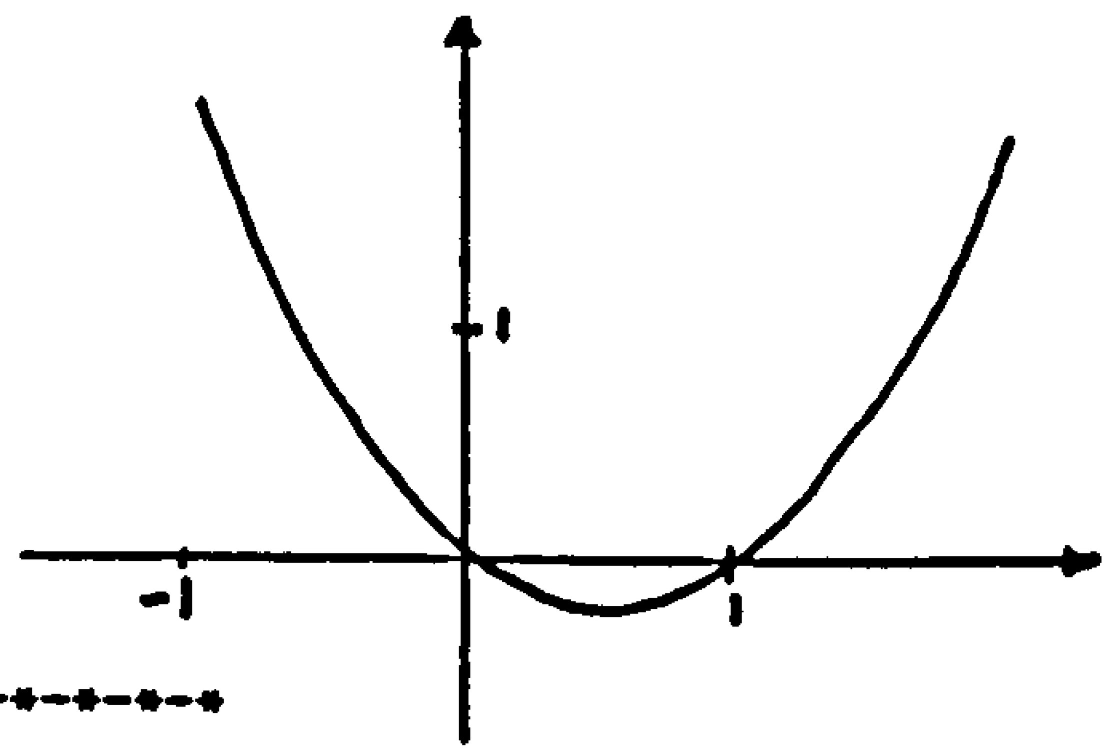
## Tangent Investigation

### THE IDEA OF A TANGENT

*This is an investigation into how students think about tangents. It is not a test. Please answer the questions carefully and try to give reasons for your answers.*

1. The graph of $y=x^2-x$

Does the graph have a tangent at x=0? YES/NO
If YES, please sketch the tangent, if NO, why not?




Are you sure of your answer?
Underline one response that bes fits your feelings:
Certain / fairly sure / fairly doubtful / very doubtful

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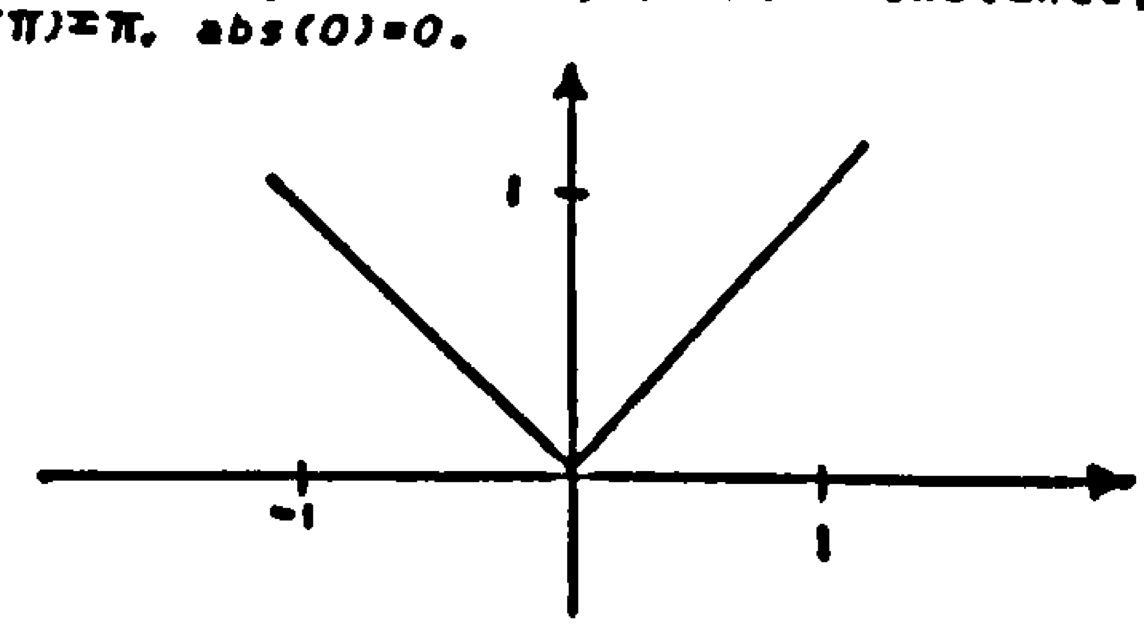
*The absolute value of a number t is denoted by abs(t) and represents the numerical value of t, regardless of sign. Thus if t is negative, the absolute value strips off the negative sign. For instance, abs(1.5)=1.5, abs(-3.7)=3.7, abs(π)=π, abs(0)=0.*

2. The graph of y=abs(x).

Does the graph have a tangent at x=0? YES/NO
If YES, please sketch the tangent, if NO, why not?




Are you sure of your answer?
Underline one response that best fits your feelings:
Certain / fairly sure / fairly doubtful / very doubtful

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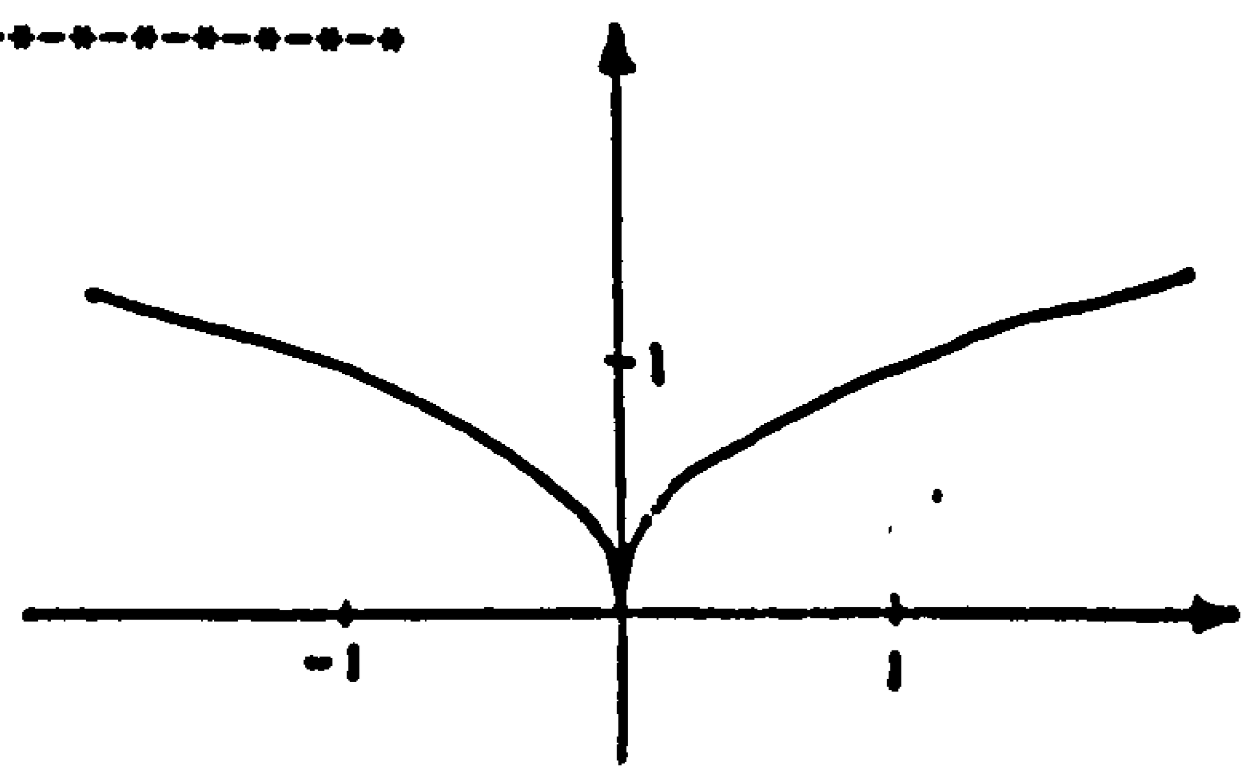
3. The graph of y=√(abs(x))

Does the graph have a tangent at x=0? YES/NO
If YES, please sketch the tangent, if NO, why not?




Are you sure of your answer?
Underline one response that best fits your feelings:
Certain / fairly sure / fairly doubtful / very doubtful

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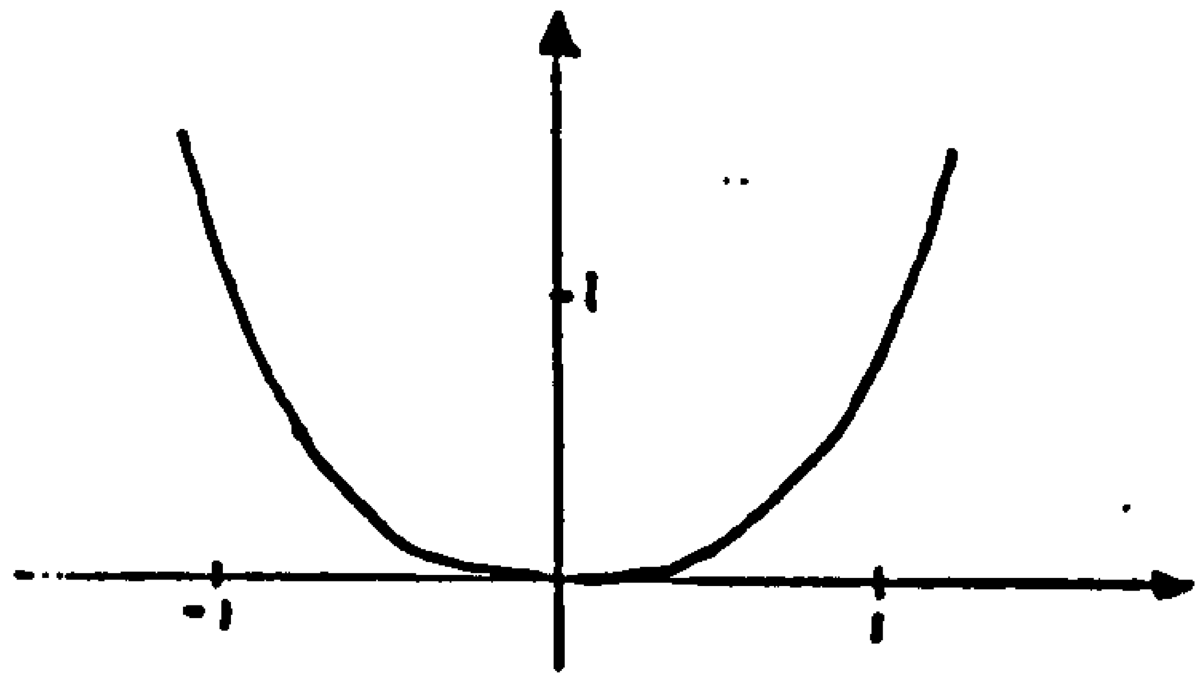
4. abs($x^3$)




Does the graph have a tangent at x=0? YES/NO
If YES, please sketch the tangent, if NO, why not?

Are you sure of your answer?
Underline one response that best fits your feelings:
Certain / fairly sure / fairly doubtful / very doubtful

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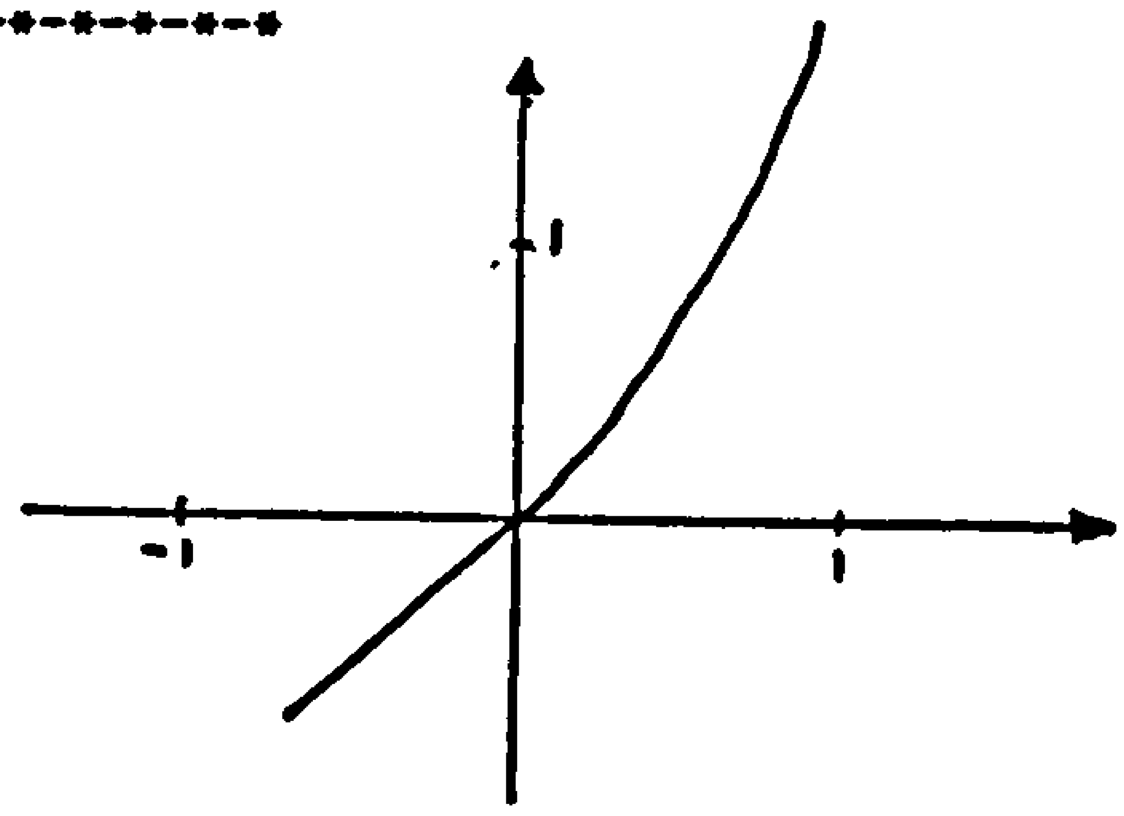
5. The graph of $y=\begin{cases} x & (x\le 0) \\ x+x^2 & (x\ge 0)\end{cases}$




Does the graph have a tangent at x=0? YES/NO
If YES, please sketch the tangent, if NO, why not?

Are you sure of your answer?
Underline one response that best fits your feelings:
Certain / fairly sure / fairly doubtful / very doubtful

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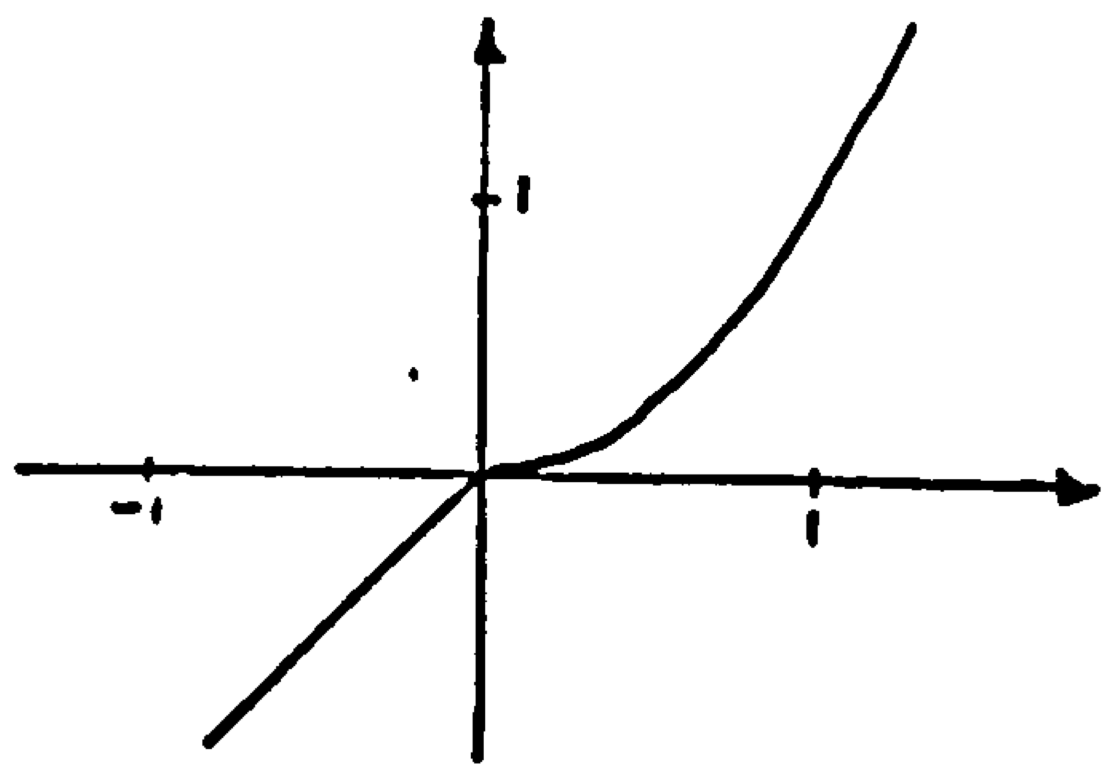
6. The graph of $y=\begin{cases} x & (x\le 0) \\ x^2 & (x\ge 0)\end{cases}$




Does the graph have a tangent at x=0? YES/NO
If YES, please sketch the tangent if NO, why not?

Are you sure of your answer?
Underline one response that best fits your feelings:
Certain / fairly sure / fairly doubtful / very doubtful

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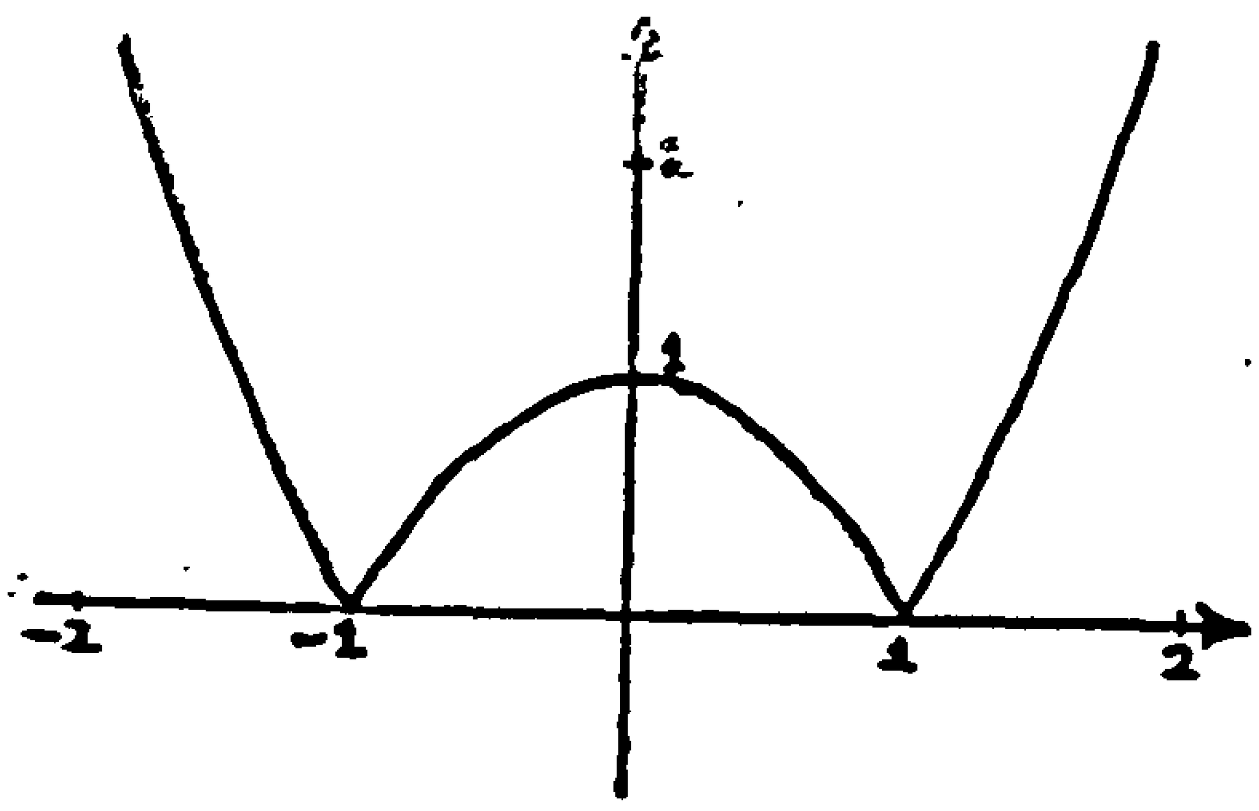
*7. The absolute value abs(t) of any number t is the numerical value of t with the sign made positive, abs(-1.5)=1.5, abs(1)=1, abs(-2)=2 etc. The displayed picture is the graph of $y=abs(x^2-1)$. It can also be expressed as: $y=x^2-1$ if $x\le-1$ or $x\ge1$ and $y=1-x^2$ for $-1\le x\le1$.*




Which of the following is true:
(a) the graph has no gradient at x=1
(b) the graph has one gradient at x=1
(c) it has two gradients at x=1
(d) it has more than two gradients at x=1
(e) other comment (specify).....

Circle one of: a - b - c - d - e
If your response is (a), say why not, if (b),(c) or (d) specify the gradient(s):

How sure are you of your answer? (underline one)
Certain/fairly sure/fairly doubtful/very doubtful.

********************************************

Which of the following are true?
(a) the graph has no derivative at x=1
(b) the graph has one derivative at x=1
(c) the graph has two derivatives at x=1
(d) the graph has more than two derivatives at x=1
(e) other comment (specify).....

Circle one of: a - b - c - d - e

If your response is (a), say why not, if (b),(c) or (d) specify the derivative(s):

How sure are you of your answer? (underline one)
Certain/fairly sure/fairly doubtful/very doubtful.

********************************************

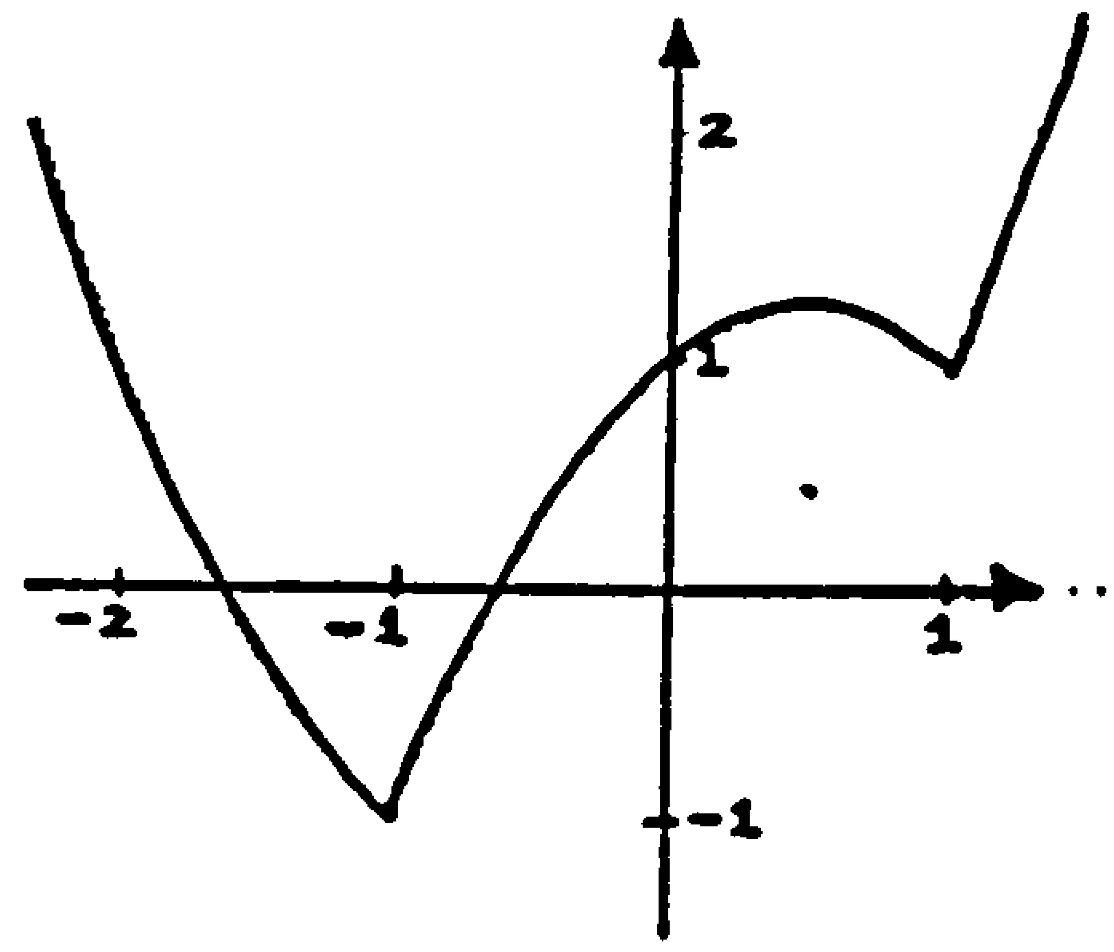
Which of the following is true:
(a) the graph has no tangent at x=1
(b) the graph has one tangent at x=1
(c) the graph has two tangents at x=1
(d) the graph has more than two tangents at x=1
(e) other comment (specify).....

Circle one of: a - b - c - d - e

If your response is (a), say why not, if (b) (c) or (d) draw the tangent(s) on the above graph...

How sure are you of your answer? (underline one)
Certain/fairly sure/fairly doubtful/very doubtful.

8. **The displayed picture is the graph of $y=abs(x^2-1)+x$. It can also be expressed as: $y=x^2-1+x$ if $x\le-1$ or $x\ge1$ and $y=1-x^2+x$ for $-1\le x\le1$.**




Which of the following is true:
(a) the graph has no gradient at x=1
(b) the graph has one gradient at x=1
(c) it has two gradients at x=1
(d) it has more than two gradients at x=1
(e) other comment (specify).....

Circle one of: a - b - c - d - e
If your response is (a), say why not, if
(b),(c) or (d) specify the gradient(s):

How sure are you of your answer? (underline one)
Certain/fairly sure/fairly doubtful/very doubtful.

##########################################
Which of the following is true?
(a) the graph has no derivative at x=1
(b) the graph has one derivative at x=1
(c) the graph has two derivatives at x=1
(d) the graph has more than two derivatives at x=1
(e) other comment (specify).....

Circle one of: a - b - c - d - e

If your response is (a), say why not, if
(b),(c) or (d) specify the derivative(s):

How sure are you of your answer? (underline one)
Certain/fairly sure/fairly doubtful/very doubtful.

##########################################
Which of the following is true:
(a) the graph has no tangent at x=1
(b) the graph has one tangent at x=1
(c) the graph has two tangents at x=1
(d) the graph has more than two tangents at x=1
(e) other comment (specify).....

Circle one of: a - b - c - d - e

If your response is (a), say why not, if (b) (c) or (d) draw the
tangent(s) on the above graph...

How sure are you of your answer? (underline one)
Certain/fairly sure/fairly doubtful/very doubtful.

##################################################

*Please write your name ................*
*Have you seen tangents drawn on the BBC computer? YES/NO*
*Have you used the computer to help answer these questions? YES/NO*

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